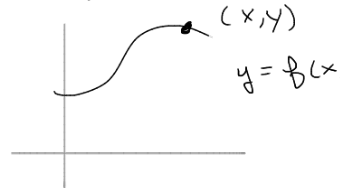
Thursday – Week 13

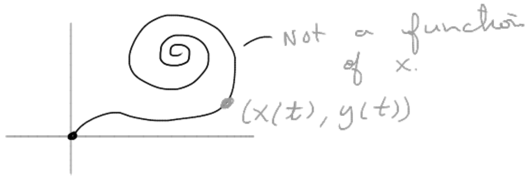
(Take-Home is up)

Arc length / Area of Parametric Curves / Surfaces

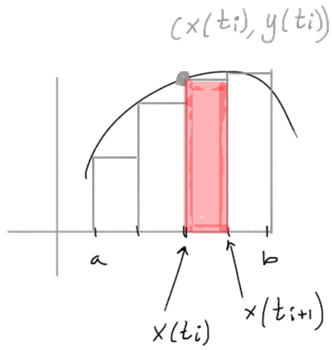
Our formulas for these work for



we can't yet compute arc length



Area



Approximate w/ rectangles ; Single Rectangle ,

$\frac{\Delta x}{\Delta t} \longrightarrow \frac{dx}{dt} = x'(t)$

Area $= l \times w = y(t_i)\cdot \Delta x_i$

$= y(t_i)\cdot \frac{(x(t_{i+1})-x(t_i))}{(t_{i+1}-t_i)} \cdot (t_{i+1}-t_i)$

width is constant :

$\Delta x_i = x(t_{i+1}) - x(t_i)$

Approx total Area $= \sum_{i=0}^{N} y(t_i)\cdot\left(\frac{x(t_{i+1})-x(t_i)}{t_{i+1}-t_i}\right)\cdot\underbrace{(t_{i+1}-t_i)}_{\Delta t_i}$

Actual Area take $\lim_{N\to\infty}$ $\Longrightarrow$ $\boxed{A = \int_a^b y(t)\cdot x'(t)\,dt}$

Area under parametric curve.

Ex. $x(t) = t - \sin(t)$ $\quad 0 \le t \le 2\pi$

$y(t) = 1 - \cos(t)$

$u = 2t$
$du = 2dt$
$\frac{1}{2}du = dt$

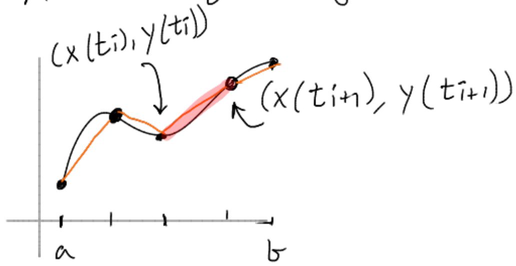
$\int \frac{1}{2}\cos(2t)\,dt$
$= \frac{1}{2}\int \cos u \,\frac{1}{2}du$

$$L = \int_0^{2\pi} (1-\cos(t))(1-\cos(t))\,dt$$

$$= \int_0^{2\pi} 1 - 2\cos(t) + \cos^2(t)\,dt = \int_0^{2\pi} 1 - 2\cos(t) + \frac{1+\cos(2t)}{2}\,dt$$

$$= t - 2\sin(t) + \frac{1}{2}t + \frac{1}{4}\sin(2t)\Big|_0^{2\pi} = 2\pi - 2\cdot 0 + \frac{1}{2}(2\pi) + \frac{1}{4}(0) = 3\pi$$

Arc length of Parametric curves



Single segment length:

$$\sqrt{\left(X(t_{i+1}) - X(t_i)\right)^2 + \left(Y(t_{i+1}) - Y(t_i)\right)^2}$$

$$\overset{\text{M.V.T}}{=} \sqrt{\left(X'(t_i^*)\Delta t_i\right)^2 + \left(Y'(t_i^*)\Delta t_i\right)^2}$$

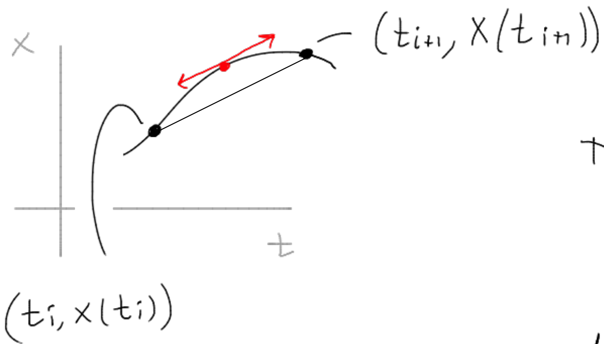
$$= \sqrt{\left(x'(t_i^*)\right)^2 + \left(y'(t_i^*)\right)^2}\;\Delta t_i$$

Take sum over all rectangles, then limit as # rectangles $\to \infty$

$$= \boxed{L = \int_a^b \sqrt{\left[X'(t)\right]^2 + \left[y'(t)\right]^2}\, dt}$$

Arc length of a parametric curve.

Consider the x-t & y-t curves



$$\text{slope of segment} = \frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i} = X'(t_i^*)$$

by Mean Value theorem

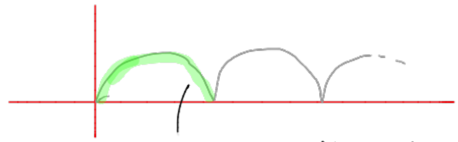
or $\quad x(t_{i+1}) - x(t_i) = x'(t_i^*) \cdot \underbrace{(t_{i+1} - t_i)}_{\Delta t_i}$

Ex

$x(t) = t - \sin(t)$

$y(t) = 1 - \cos(t)$

$0 \le t \le 2\pi$



$$L = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(1-\cos(t))^2 + (\sin(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos(t) + \underbrace{\cos^2(t) + \sin^2(t)}_{=1}}\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos(t)}$$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos(t)}\, dt$$

Your problem on take Home:

$x = \sin^3(t)$

$y = \cos^3(t)$

stop here

$\int_0^{2\pi}$ ... you can solve the integral by hand.