Name:
Math 163 Final Exam Guide
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Integration

1.

$$\int 4x^3 \cos(x^4-5)dx$$

$u = x^4 - 5$
$du = 4x^3 dx$

$$= \int \cos(u)\,du = \sin(u) + C$$
$$= \sin(x^4-5) + C$$

2.

$$\int x \sin x dx$$

$u = x \quad dv = \sin x\, dv$
$du = dx \quad v = -\cos x$

$$= -x\cos x + \int \cos x\, dx = \boxed{-x\cos x + \sin x + C}$$

3.

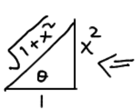
$$\int \frac{4x-5}{\underbrace{x^2-7x-8}_{(x-7)(x-1)}}dx = \int \frac{A}{x-7} + \frac{B}{x-1}\,dx$$

$4x-5 = A(x-1) + B(x-7)$
$x=1 \Rightarrow -1 = B(-6) \quad B = 1/6$
$x=7 \Rightarrow 23 = A(6)$
$A = 23/6$

$$= \frac{23}{6}\int \frac{1}{x-7}dx + \frac{1}{6}\int\frac{1}{x-1}dx$$
$$= \boxed{\frac{23}{6}\ln|x-7| + \frac{1}{6}\ln|x-1| + C}$$

4. See $1+x^2$:

$$\int \frac{x^3}{\sqrt{1+x^2}}dx$$



$x = \tan\theta$, $\theta = \tan^{-1}x$
$dx = \sec^2\theta\, d\theta$

start

$$= \int \frac{\tan^3\theta}{\sqrt{\underbrace{1+\tan^2\theta}_{\sec^2\theta}}}\cdot \sec^2\theta\, d\theta$$

$$= \int \tan^3\theta \cdot \sec\theta\, d\theta = \int \tan\theta(\sec^2\theta - 1)\sec^2\theta\, d\theta$$
$$= \int \sec^4\theta\tan\theta - \sec^2\theta\tan\theta\, d\theta$$
$$= \int \sec^3\theta \sec\theta\tan\theta\, d\theta - \int \sec\theta\sec\theta\tan\theta\, d\theta$$

$u = \sec\theta$
$du = \sec\theta\tan\theta$

$$= \int u^3 - u\, du = \frac{u^4}{4} - \frac{u^2}{2} + C = \frac{\sec^4\theta}{4} - \frac{\sec^2\theta}{2} + C$$

$\sec\theta = \sqrt{1+x^2}$ $\Longrightarrow$

$$= \frac{(1+x^2)^2}{4} - \frac{(1+x^2)}{2} + C$$

5.

$$\int \tan^3(\theta)\sec(\theta)d\theta$$

$\frac{\sec^4\theta}{4} - \frac{\sec^2\theta}{2} + C$

see previous exercise

6.

$$\int \sin^{-1}(x)dx$$

$u = \sin^{-1}x \quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} \quad v = x$

$= x\sin^{-1}x - \frac{1}{2}\int \frac{2x}{\sqrt{1-x^2}}dx \qquad u = 1-x^2, \quad du = -2x\,dx$

$= x\sin^{-1}x - \frac{1}{2}\int \frac{du}{\sqrt{u}}$

$= x\sin^{-1}x - \frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) = \boxed{x\sin^{-1}x - \sqrt{1-x^2} + c}$

7.

$$\int_0^{+\infty} e^{-4x}dx$$

$= \lim_{N\to\infty}\int_0^N e^{-4x}dx = \lim_{N\to\infty} -\frac{1}{4}e^{-4x}\Big|_0^N = \lim_{N\to\infty} -\frac{1}{4}\left(\frac{1}{e^{4N}} - 1\right) = \frac{1}{4}$

8.

$$\int e^{4x}\sin x dx \qquad \frac{16}{17}\left[-e^{4x}\cos x + \frac{1}{4}e^{4x}\sin x\right] + c$$

$u = e^{4x} \quad dv = \sin x\,dx$

$du\ \frac{1}{4}e^{4x} \quad V = -\cos x$

$= -e^{4x}\cos x + \frac{1}{4}\int e^{4x}\cos x = -e^{4x}\cos x + \frac{1}{4}\left[e^{4x}\sin x - \frac{1}{4}\int e^{4x}\sin x\,dx\right]$

$u = e^{4x} \quad dv = \cos x$

$du = \frac{1}{4}e^{4x} \quad v = \sin x$

$= -e^{4x}\cos x + \frac{1}{4}e^{4x}\sin x - \frac{1}{16}\underbrace{\int e^{4x}\sin x\,dx}_{\text{isolate}}]$

$\Rightarrow$

9.

see $4-x^2$:

$x = 2\sin\theta$

$dx = 4\cos\theta\, d\theta$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\sqrt{1-\sin^2\theta} = 2\cos\theta$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\sin\theta = \frac{x}{2}$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$= 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2}$

$= \frac{x\sqrt{4-x^2}}{2}$

$$\int \sqrt{4-x^2}\,dx = 2\int \cos\theta \cdot 4\cos\theta\, d\theta$$

$$= 8\int \cos^2\theta\, d\theta$$

$$= 8\int \frac{1}{2}(1+\cos(2\theta))\,d\theta$$

$$= 4\int 1\,d\theta + \frac{4}{2}\int \cos(u)\,du \qquad u = 2\theta \quad du = 2\,d\theta$$

$$= 4\theta + 2\sin(u) + C$$

$$= 4\sin^{-1}\left(\frac{x}{2}\right) + 2\sin(2\theta) + C$$

$$= 4\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C$$

10.

$$\int \sec^3(x)\,dx$$

$$\int \sec^3(x)\,dx \overset{IBP}{=} \int \sec^2 x \cdot \sec x\,dx = \sec x\tan x - \int \sec x\tan^2 x\,dx$$

$dv = \sec^2 x$, $v = \tan x$; $u = \sec x$, $du = \sec x\tan x$

$$\int \sec^5(x)\,dx$$

$$= \sec x\tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec\tan x - \int \sec^3 x - \int \sec x$$

add integral of sec^3(x) to both sides

$$2\int \sec^3(x)\,dx = \sec\tan x - \int \sec x$$

divide by 2

$$\int \sec^3(x)\,dx = \frac{\sec\tan x - \int \sec x}{2} = \frac{\sec\tan x - \ln|\sec x + \tan x|}{2} + C$$

Sequences & Series

11. Determine whether the series converges or diverges:

(11.1)

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

diverges, p test

(11.2)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges

by A.S.T.

since $\{\frac{1}{n}\}$ is decreasing

(11.3)

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

$\frac{1}{n!} < \frac{1}{n^2}$ for $n > 3$

converges by D.C.T.

(11.4)

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum \left(\frac{2}{3}\right)^n$$

converges to $\frac{1}{1-\frac{2}{3}} = 3$

by Geometric series Test

(11.5)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

A.S.T. $\Rightarrow$ converges

$\frac{1}{\sqrt{n}}$ is decreasing

(11.6) $\frac{\sqrt{k}}{k} = \frac{1}{k^{2/3}}$

$$\sum_{k=1}^{\infty} \frac{\sqrt{k}}{3k+4}$$

L.C.T. $\frac{\frac{\sqrt{k}}{3k+4}}{\frac{1}{k^{2/3}}} = \frac{\sqrt{k}}{3k+4} \cdot \frac{k^{2/3}}{1} = \frac{k}{3k+4} \xrightarrow{k\to\infty} \frac{1}{3}$ $\quad 0 < \frac{1}{3} < \infty$

Converges

(11.7)

$$\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{4k^2+9}$$

$\frac{\sqrt[3]{k}}{4k^2+9} < \frac{\sqrt[3]{k}}{4k^2} = \frac{1}{4k^{5/3}} = \frac{1}{4}\left(\frac{1}{k^{5/3}}\right) \to$ converges by p-test

converges by D.S.T.

(11.8)

$$\sum_{k=1}^{\infty} \frac{k!}{2^k}$$

$\left| \frac{\frac{(k+1)!}{2^{k+1}}}{\frac{k!}{2^k}} \right| = \left| \frac{(k+1)!}{2^{k+1}} \cdot \frac{2^k}{k!} \right| = \frac{k+1}{2} \longrightarrow \infty$ $\quad$ diverges by ratio test

(11.9)

$$\sum_{k=1}^{\infty} \sqrt{\frac{25k}{100+k}}$$

diverges by Divergence Test

$\lim_{k\to\infty} \sqrt{\frac{25k}{100+k}} = 5 \neq 0$

(11.10)

$$\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt[3]{k}}{4k^2+9}$$

$\frac{\sqrt[3]{k}}{4k^2+9}$ is decreasing since

if $f(x) = \frac{\sqrt[3]{x}}{4x^2+9} \Rightarrow f'(x) = \frac{(4x^2+9)\frac{1}{3}x^{-\frac{2}{3}} - \sqrt[3]{x}\cdot 8x}{(4x^2+9)^2} < 0$ for $x > 1$

(11.11)

$$\sum_{k=1}^{\infty} \frac{9}{2^k}$$

$\frac{9}{2^k} < \frac{9}{k^2}$ for $k > 4$

↳ p-series test ; converges

↳ converges by D.C.T.

(11.12)

$$\sum_{k=1}^{\infty}[64^{1/k} - 64^{1/(k+2)}]$$

$S_1 = 64 - 64^{1/3}$

$S_2 = 64 - 64^{1/3} + 64^{1/2} - 64^{1/4}$

$S_3 = 64 - 64^{1/3} + 64^{1/2} - 64^{1/4} + 64^{1/3} - 64^{1/5} = 64 + 8 - 64^{1/4} - 64^{1/5}$

$S_4 = 64 - 64^{1/3} + 64^{1/2} - 64^{1/4} + 64^{1/3} - 64^{1/5} + 64^{1/4} - 64^{1/6} = 72 - 64^{1/5} - 64^{1/6}$

$\vdots$

$S_n = 72 - 64^{1/n} - 64^{1/n+2}$

$S = \lim_{n \to \infty} 72 - 64^{1/n} - 64^{1/n+2} = 72$

(11.13) Give three examples of a *sequence* that converges to $\ln(5)$.

$\{\ln(5) + \frac{1}{n}\}$

$\{\frac{n}{n+1} \cdot \ln(5)\}$

$\{\ln(5 + \frac{1}{n})\}$

(11.14) Give three examples of a divergent sequence.

$\{(-1)^n\}$

$\{n\}$

$\{\frac{1}{n}\}$

Taylor & Maclaurin

1. Taylor:

(1.1) Find fourth degree Taylor polynomial for the function

$$f(x) = \ln(x-1) \text{ at } x = 2$$

$f^n(x)$	$f^n(2)$	$\frac{f^n(2)}{n!}$
$\ln(x-1)$	$\ln(1)=0$	0
$\frac{1}{x-1}$	1	1
$\frac{-1}{(x-1)^2}$	-1	$-\frac{1}{2}$
$\frac{2}{(x-1)^3}$	2	$\frac{2}{3}$
$\frac{-6}{(x-1)^4}$	-6	$\frac{-6}{6}=-1$

$$= (x-2) - \tfrac{1}{2}(x-2)^2 + \tfrac{2}{3}(x-2)^3 - (x-2)^4$$

(1.2) Find the interval of convergece for the power series below:

$$\gamma(x) = x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} x^{3n+1}$$

Endpts!

$x=-1 \Rightarrow \frac{(-1)^n(-1)^{3n+1}}{3n+1} = \frac{(-1)^{4n+1}}{3n+1}$ ← ALTERNATE $\Rightarrow$ Conv. by A.S.T.

$x=1 \Rightarrow \frac{(-1)^n}{3n+1}$ Conv. by AST

Ratio Test

& abs. value

$$\frac{\frac{x^{3(n+1)+1}}{3(n+1)+1}}{\frac{x^{3n+1}}{3n+1}} = \frac{x^{\overbrace{3(n+1)+1}^{3n+3+1}}}{3(n+1)+1} \cdot \frac{3n+1}{x^{3n+1}} = \frac{x^{\overbrace{3n+4}^{x^{3n}\cdot x^4}}}{3n+4} \cdot \frac{3n+1}{\underbrace{x^{3n+1}}_{x^{3n}\cdot x}} = \frac{x^3 \cdot 3n+1}{3n+4} \to |x^3| < 1$$

$\Rightarrow$ $\boxed{x \in [-1,1]}$

(1.3) Using the $\gamma(x)$ from the previous problem find the limit

$$\lim_{x\to 0} \frac{\gamma(x) - x}{x^4} = \lim_{x\to 0} \frac{x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \ldots - x}{x^4}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{4}x^4 + \frac{1}{7}x^7 - \ldots}{x^4} = \lim_{x\to 0} \frac{x^4\left[-\frac{1}{4} + \frac{1}{7}x^3 - \ldots\right]}{x^4} = \lim_{x\to 0}\left[-\frac{1}{4} + \frac{1}{7}x^3 - \ldots\right]$$

$$= \boxed{-\frac{1}{4}}$$

(1.4) Use a seventh degree polynomial to estimate

$$\int_0^1 \gamma(x)dx$$

$$\int_0^1 x - \frac{1}{4}x^4 + \frac{1}{7}x^7 \, dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{56} = \text{0.467}$$

(1.5) Show that if $\sum_{n=0}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Assume $\sum_{n=0}^{\infty} a_n$ is convergent

"capitol N"

then $S = \sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N$ where $S_N = \sum_{n=0}^{N} a_n$.

Since $a_n = S_n - S_{n-1}$ it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1} = \underbrace{\lim_{n\to\infty} S_n}_{=S} - \underbrace{\lim_{n\to\infty} S_{n-1}}_{=S} = 0$$

(1.6) Find the Maclaurin series representation of the function $f(x) = e^x$. (Show your work, i.e., derive the series from scratch)

n	$f^n(x)$	$f^n(0)$	$\frac{f^n(0)}{n!}$
0	e^x	1	1
1	e^x	1	1
2	e^x	1	$\frac{1}{2}$
3	e^x	1	$\frac{1}{3!}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

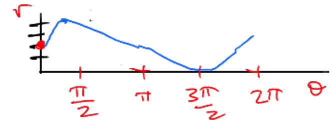
$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$$

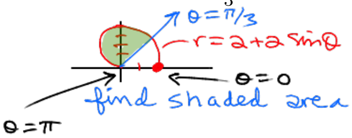
$$= \sum \frac{x^n}{n!}$$

Parametric, Polar & Differential Equations

5. Find the area of the region bounded by the curve $r = 2 + 2\sin\theta$ and the line $\theta = \frac{\pi}{3}$.

$y = 2 + 2\sin x = 2(1+\sin x)$





$$\int_{\frac{\pi}{3}}^{\frac{3\pi}{2}} 2 + 2\sin\theta \, d\theta = 2\theta - 2\cos\theta \Big|_{\frac{\pi}{3}}^{\frac{3\pi}{2}} = 2\tfrac{3\pi}{2} - 2\cos\tfrac{3\pi}{2} - \left(2\tfrac{\pi}{3} - 2\cos\tfrac{\pi}{3}\right)$$

$$\boxed{\frac{5\pi}{3} - 1}$$

6. Compute the arc length of a circle centered at the origin of radius 4 using polar coordintes.

Polar Arc length Formula $= \int_a^b \sqrt{(f(\theta))^2 + (f'(\theta))^2} \, d\theta$ $\quad \rightarrow r = 4$, $f(\theta) = 4$

Circle $\Rightarrow$ $a = 0$, $b = 2\pi$, $f(\theta) = 4$, $f'(\theta) = 0$

$$= \int_0^{2\pi} \sqrt{4^2 + 0^2} \, d\theta = 4\int_0^{2\pi} d\theta = 4\theta \Big|_0^{2\pi} = 8\pi$$

Geometry

$C = 2\pi r$

$= 2\pi \cdot 4 = 8\pi$

check:

$y' = -2\ln|x| - 2x\left(\frac{1}{x}\right) + \frac{\pi}{2} + 2\ln(2)$

$y' \cdot x = -2x\ln|x| - 2x + \left[\frac{\pi}{2} + 2\ln(2)\right]x$

$y - 2x$ = exactly this ;

$y(2) = -2 \cdot 2 \cdot \ln(2) + \left[\frac{\pi}{2} + 2 \cdot \ln(2)\right] 2 = \pi$

☺

7. Solve the following initial value problem:

$$x\frac{dy}{dx} = y - 2x, y(2) = \pi$$

$\div$ by x : $\quad y' = \frac{y}{x} - 2$

std. Form : $\quad y' - \frac{1}{x}y = -2$

I.F. $\quad e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = e^{\ln x^{-1}} = x^{-1}$

$y' \cdot x^{-1} - \frac{1}{x}(x^{-1})y = -2x^{-1}$

$\underbrace{y' \cdot x^{-1} - x^{-2}y}_{\frac{d}{dx}(y \cdot x^{-1})} = -2x^{-1}$

integrate both sides :

$y \cdot x^{-1} = \int -2x^{-1}\,dx = -2\int \frac{1}{x}\,dx$

$= -2\ln|x| + C$

$y = -2x \cdot \ln|x| + Cx$

$y(2) = -2 \cdot 2 \cdot \ln(2) + C \cdot 2 = \pi$

$C = \frac{\pi}{2} + 2\ln(2)$

$$\boxed{y = -2x\ln|x| + \left[\frac{\pi}{2} + 2\ln(2)\right]x}$$

8. Solve the following initial value problem:

should be e^{-4x}

$$\frac{dy}{dx} + 4y = e^{-4t}, y(0) = 4$$

I.F. $e^{\int 4dx} = e^{4x}$ multiply through

$\underbrace{y' \cdot e^{4x} + e^{4x} \cdot 4 \cdot y}_{\frac{d}{dx}(ye^{4x})} = 1$

integrate $\Rightarrow$

$ye^{4x} = \int 1\, dx = x + C$

$y = xe^{-4x} + Ce^{-4x}$

$y(0) = Ce^{0} = 4 \Rightarrow C = 4$

$y = xe^{-4x} + 4e^{-4x} = (x+4)e^{-4x}$

check

$y' = e^{-4x} + (x+4)(-4)e^{-4x}$

$y' + 4y = e^{-4x} + (x+4)(-4)e^{-4x} + 4(x+4)e^{-4x} = e^{-4x}$

$y(0) = 4$

9. Solve the following initial value problem:

separable

$$yy' = xe^{-y^2}, y(0) = -5$$

$y' = \frac{dy}{dx}$

$y \cdot \frac{dy}{dx} = xe^{-y^2} \Rightarrow ye^{y^2}\, dy = x\, dx$

integrate

$\int ye^{y^2}\, dy = \int x\, dx$

$\frac{1}{2}e^{y^2} = \frac{1}{2}x^2 + C$

$e^{y^2} = x^2 + C$

$y^2 = \ln(x^2 + C)$

$y(0) = -5 \Rightarrow$

$25 = \ln(C)$, $C = e^{25}$

$\boxed{y^2 = \ln(x^2 + e^{25})}$

10. Solve the following initial value problem:

$$(1 - 9t)\frac{dy}{dt} - y = 0, y(2) = -6$$

$e^{A+B} = e^{A} \cdot e^{B}$

$(1-9t)\frac{dy}{dt} = y$

$(1-9t)dy = y\, dt$

$\frac{1}{y}dy = \frac{1}{1-9t}dt$ $\quad u = 1-9t$, $du = -9dt$

$\ln|y| = -\frac{1}{9}\ln|1-9t| + C = \ln|1-9t|^{-1/9} + C$

exponentiate both sides

$|y| = e^{|1-9t|^{-1/9} + C} = |1-9t|^{-1/9} \cdot \underbrace{e^{C}}_{c_1}$

$|y(2)| = |1-18|^{-1/9} \cdot c_1 \Rightarrow -6 = (-17)^{-1/9} \cdot c_1 \Rightarrow c_1 = 6 \cdot 17^{1/9}$

$\boxed{y = \frac{6(17)^{1/9}}{(1-9t)^{1/9}}}$

note: $y(2) = -6$