some standard trig substitution problems

1. Use $x^2 = 4\sin^2\theta \implies x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2\theta}\,(2\cos\theta)\;d\theta$$

$$= \int \sqrt{4\cos^2\theta}\,(2\cos\theta)\;d\theta = \int 4\cos^2\theta\,d\theta$$

$$= \int 4\cdot\frac{1}{2}\,(1+\cos 2\theta)\;d\theta = \int 2+2\cos 2\theta\,d\theta$$

$$= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C$$

2. Use $4x^2 = \tan^2\theta \implies 2x = \tan\theta$, then $dx = \frac{1}{2}\sec^2\theta\,d\theta$.

$$\int \frac{1}{\sqrt{4x^2+1}}\,dx = \int \frac{1}{\sqrt{\tan^2\theta+1}}\cdot\frac{1}{2}\sec^2\theta\,d\theta$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta = \frac{1}{2}\int \sec\theta\,d\theta$$

$$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{2}\ln|2x + \sqrt{4x^2+1}| + C$$

3. Use $x^2 = 9\sec^2\theta \Longrightarrow x = 3\sec\theta$, then $dx = 3\sec\theta\tan\theta\, d\theta$.

$$\int \frac{1}{\sqrt{x^2-9}}\, dx = \int \frac{1}{\sqrt{9\sec^2\theta - 9}}\,(3\sec\theta\tan\theta)\ d\theta$$

$$= \int \frac{1}{\sqrt{9\tan^2\theta}}\,(3\sec\theta\tan\theta)\ d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{x}{3} + \frac{\sqrt{9-x^2}}{3}\right| + C$$

4. Use $x^2 = 9\sin^2\theta \Longrightarrow x = 3\sin\theta$, then $dx = 3\cos\theta$.

$$\int \frac{1}{\sqrt{9-x^2}}\, dx = \int \frac{1}{\sqrt{9-9\sin^2\theta}}\,(3\cos\theta)\ d\theta$$

$$= \int \frac{1}{\sqrt{9\cos^2\theta}}\,(3\cos\theta)\ d\theta = \int 1\, d\theta$$

$$\theta + C = \sin^{-1}\left(\frac{x}{3}\right) + C$$