trig substitution guide

Signs that a trigonometric substitution may be useful - we're looking for patterns $\# - \text{var}^2$, $\text{var}^2 - \#$, or $\text{var}^2 - \#$, usually (but not always) under a radical.

The idea is to replace the pattern with a perfect square, allowing us to process the square root eifectly and proceed with the integral. We can use the Pythagorean trigonometric identities to accomplish this.

Pattern		Match		
		$1 - \sin^2\theta$	$=$	$\cos^2\theta$
$\# - \text{var}^2$	$=$	$\# - \#\sin^2\theta$	$=$	$\#\cos^2\theta$
		$\sec^2\theta - 1$	$=$	$\tan^2\theta$
$\text{var}^2 - \#$	$=$	$\#\sec^2\theta - \#$	$=$	$\#\tan^2\theta$
		$\tan^2\theta + 1$	$=$	$\sec^2\theta$
$\text{var}^2 + \#$	$=$	$\#\tan^2\theta + \#$	$=$	$\#\sec^2\theta$

Example:

$$\int \frac{\sqrt{9-4x^2}}{x^2}\,dx$$

The pattern is $\# - \text{var}^2$, and we want to match the 9, not the 4. (Variables are flexible and can "absorb" coeffiecients, but constants are not flexible.)

$$\begin{aligned} & & 1 - \sin^2\theta &= \cos^2\theta \\ 9 - 4x^2 &= & 9 - 9\sin^2\theta &= 9\cos^2\theta \end{aligned}$$

Note that the first and thrid parts of the equation above give us a direct substitution for $9 - 4x^2$. We can replace it with $9\cos^2\theta$.

"Analyze" the first and second parts of this equality:

$$9 - 4x^2 = 9 - 9\sin^2\theta \Longrightarrow 9\sin^2\theta = 4x^2 \Longrightarrow 2x = 3\sin\theta$$

Solve for x and differentiate to get the substitution and the "exchange rate" for dx:

$$x = \frac{3}{2}\sin\theta \Longrightarrow dx = \frac{3}{2}\cos\theta\, d\theta$$

Isolate the trig term to get to the triangle in terms of θ:

$$\frac{2x}{3} = \sin\theta \to \text{ hyp } = 3, \text{ opp } = 2x \to \text{adj } = \sqrt{9-4x^2}$$

Back to the integral:

$$\int \frac{\sqrt{9-4x^2}}{x^2}\, dx = \int \frac{\sqrt{9\cos^2\theta}}{\left(\frac{3}{2}\sin\theta\right)^2} \cdot \left(\frac{3}{2}\cos\theta\right)\, d\theta$$

$$= \int \frac{2\cos^2\theta}{\sin^2\theta}\, d\theta = \int 2\cot^2\theta\, d\theta = \int 2(\csc^2\theta - 1)\, d\theta$$

$$= -2\cot\theta - 2\theta + C$$

Based on the triangle we got above, $cot\theta = \frac{\sqrt{9-4x^2}}{2x}$.
Also from above, $\frac{2x}{3} = \sin\theta \to \theta = \sin^{-1}\left(\frac{2x}{3}\right)$.

$$\int \frac{\sqrt{9-4x^2}}{x^2}\, dx = -2\left(\frac{\sqrt{9-4x^2}}{2x}\right) - 2\sin^{-1}\left(\frac{2x}{3}\right) + C$$

$$= -\frac{\sqrt{9-4x^2}}{x} - 2\sin^{-1}\left(\frac{2x}{3}\right) + C$$

Check:

$$\frac{d}{dx}\left[-\frac{\sqrt{9-4x^2}}{x} - 2\sin^{-1}\left(\frac{2x}{3}\right)\right]$$

$$= \frac{d}{dx}\left[-(9-4x^2)^{1/2}x^{-1}\right] - 2\left(\frac{1}{\sqrt{1-\left(\frac{2x}{3}\right)^2}}\right) \cdot \frac{d}{dx}\left(\frac{2x}{3}\right)$$

$$= -\frac{1}{2}(9-4x^2)^{-1/2}(-8x)x^{-1} + (9-4x^2)^{1/2}x^{-2} - \frac{4}{3\sqrt{1-\frac{4x^2}{9}}}$$

$$= \frac{4}{\sqrt{9-4x^2}} + \frac{\sqrt{9-4x^2}}{x^2} - \frac{4}{\sqrt{9-4x^2}} = \frac{\sqrt{9-4x^2}}{x^2}$$