partial fractions

Use the method of partial fractions (and polynomial division) to evaluate these integrals:

1.

$$\int \frac{1-x}{x^2+3x+2}\,dx$$

work to pull apart original rational expression:

$$\frac{1-x}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1} = \frac{A(x+1)}{(x+2)(x+1)} + \frac{B(x+2)}{(x+1)(x+2)}$$

$$\Longrightarrow 1-x = A(x+1) + B(x+2)$$

When $x = -1$, $1-(-1) = A(-1+1) + B(-1+2) \Longrightarrow 2 = B$.
When $x = -2$, $1-(-2) = A(-2+1) + B(-2+2) \Longrightarrow -3 = A$.

$$\int \frac{1-x}{x^2+3x+2}\,dx = \int \frac{2}{x+1} + \frac{-3}{x+2}\,dx$$
$$= 2\ln|x+1| - 3\ln|x+2| + C$$

2. Use your answer from #1 after getting down to a proper rational function.

$$\int \frac{x^3+4x^2+4x+3}{x^2+3x+2}\,dx = \int x+1+\frac{1-x}{x^2+3x+2}\,dx$$

$$= \frac{1}{2}x^2 + x + 2\ln|x+1| - 3\ln|x+2| + C$$

3.

$$\int \frac{x^3+1}{x^3-x^2}\,dx = \int 1 + \frac{x^2+1}{x^3-x^2}\,dx$$

work to pull apart the proper rational function:

$$\frac{x^2+1}{x^2(x-1)} = \frac{Ax+B}{x^2} + \frac{C}{x-1} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$\frac{x^2+1}{x^2(x-1)} = \frac{Ax(x-1)}{x^2(x-1)} + \frac{B(x-1)}{x^2(x-1)} + \frac{Cx^2}{(x-1)x^2}$$

$$\implies x^2+1 = Ax(x-1) + B(x-1) + Cx^2$$

We can plug in values for x, but in this case it's actually easier / faster to collect terms.

$$x^2 + 0x + 1 = (A+C)x^2 + (-A+B)x - B$$

$$\implies -B = 1, -A+B = 0, \text{ and } A+C = 1$$

$$\implies B = -1, A = -1, C = 2$$

$$\int \frac{x^3+1}{x^3-x^2}\,dx = \int 1 + \frac{x^2+1}{x^3-x^2}\,dx$$

$$= \int 1 + \frac{2}{x-1} - \frac{1}{x} - \frac{1}{x^2}\,dx$$

$$= x + 2\ln|x-1| - \ln|x| + \frac{1}{x} + C$$

4.

$$\int \frac{x-2}{x^3+x}\,dx$$

work to pull apart the proper rational function:

$$\frac{x-2}{x(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x} = \frac{(Ax+B)x}{(x^2+1)x} + \frac{C(x^2+1)}{x(x^2+1)}$$

$$\Longrightarrow 0x^2 + x - 2 = (A+C)x^2 + Bx + C$$

$$\Longrightarrow A + C = 0, B = 1, \text{ and } C = -2 \Longrightarrow A = 2$$

$$\int \frac{x-2}{x^3+x}\,dx = \int \frac{2x+1}{x^2+1} + \frac{-2}{x}\,dx$$

$$= \int \frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x}\,dx$$

$$= \ln\left(x^2+1\right) + \tan^{-1} x - 2\ln|x| + C$$