Partial Fractions Technique of Integration // useful for integration Rational Functions

$$\int \frac{1}{x^2 - 7x + 12}\,dx = \int \frac{1}{(x-3)(x-4)}\,dx$$

Idea: Constants A, B exist s.t. :

$$\frac{1}{(x-3)(x-4)} \overset{\star}{=} \underbrace{\frac{A}{(x-3)} + \frac{B}{x-4}}_{\text{Find Common Denom}} = \frac{A(x-4) + B(x-3)}{(x-3)(x-4)}$$

set numerators equal: $1 = Ax - 4A + Bx - 3B$

$$1 = (A+B)x + (-4A - 3B)$$

"Equate Coefficients"

- $A + B = 0$

substitute $A = -B$

- $1 = -4A - 3B$
- $1 = -4(-B) - 3B = 4B - 3B = B$

$$\boxed{\begin{matrix} A = -1 \\ B = 1 \end{matrix}}$$

$$\text{Ans} \overset{\star}{=} \int \frac{-1}{x-3} + \frac{1}{x-4}\,dx$$

$$= \boxed{-\ln|x-3| + \ln|x-4| + C}$$

Variations —

Repeated Factors

$u = x+1 \qquad = \int \frac{-1}{u^2}\,du = -\int u^{-2}\,du$

$du = dx \qquad\qquad\qquad\qquad\quad = \frac{-u^{-1}}{-1} + c$

$$\int \frac{1}{x(x+1)^2}\,dx = \int \frac{1}{x} + \frac{-1}{x+1} + \frac{-1}{(x+1)^2}\,dx = \ln|x| - \ln|x+1| + \frac{1}{x+1} + C$$

Idea:

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

Find A, B, C neccessary for this eqn to be true (common denom.)

numerators: $1 = A(x+1)^2 + Bx(x+1) + Cx$

$= Ax^2 + 2Ax + A + Bx^2 + Bx + Cx$

$1 = (A+B)x^2 + (2A + B + C)x + A$

set $x = 0$: $1 = A$

$0 = A + B$ (no x^2 term on LHS)

$B = -1$

Plug these in to find C:

(No linear term on LHS) $\Rightarrow$ $2A + B + C = 0$

$2(1) + (-1) + C = 0$

$C = -1$

Variation:

Irreducible Quadratic:

Fact: x^2+1 is irreducible. (no real roots) / doesn't factor of the integers

$(x-i)(x+i)$

$x^2 \underbrace{+ix-ix}_{0} -i^2 = x^2+1$

$-(\sqrt{-1})^2$

$$\int \frac{1}{x(x^2+1)}\,dx = \int \frac{1}{x} + \frac{-x}{x^2+1}\,dx = \boxed{\ln|x| - \tfrac{1}{2}\ln|x^2+1| + C}$$

$u = x^2+1$, $du = 2x\,dx$ | $dx = \frac{1}{2x}du$ $\Big\} = \int \frac{-x}{u} \cdot \frac{1}{2x}\,du$

Idea:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \quad \text{So...} \quad 1 = A(x^2+1) + x(Bx+C)$$

$= Ax^2 + A + Bx^2 + Cx$

$1 = (A+B)x^2 + Cx + A$

$A+B=0$

$C=0$

$A=1$

so $B=-1$

verify:

$$\frac{1}{x} + \frac{-x}{x^2+1} = \frac{x^2+1 + -x^2}{x(x^2+1)} = \frac{1}{x(x^2+1)}$$

Lastly, Partial Fractions "work" only when

$$\int \frac{P(x)}{Q(x)}\,dx \qquad \deg(P(x)) < \deg(Q(x))$$

——— Long division may be needed ...

$$\frac{2}{(x-1)} + \frac{2}{x+1} = 2\ln|x-1| + 2\ln|x+1|$$

Ex

$$\int \frac{5x^3 + 4x^2 - x - 4}{x^2 - 1}\,dx = \int 5x + 4 + \frac{4x}{x^2-1}\,dx$$

$$= \frac{5x^2}{2} + 4x + 2\ln|x^2-1|$$

Long Division:

$$\begin{array}{r|l} & 5x + 4 \\ \hline x^2 - 1 & 5x^3 + 4x^2 - x - 4 \\ & -(5x^3 \qquad - 5x) \\ \hline & \qquad 4x^2 + 4x - 4 \\ & \qquad -(4x^2 \qquad -4) \\ \hline & \qquad\quad 4x \end{array}$$

Side

$$2\int \frac{2x}{x^2-1}\,dx \qquad u = x^2 - 1, \quad du = 2x\,dx$$

$$= 2\int \frac{du}{u} = 2\ln|u|$$

$$= 2\ln|x^2-1|$$