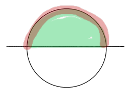
wk 3 - Mon

7-3 . Trig Substitution

useful for these types of integrals  "area of circle"

$$\int \sqrt{1-x^2}\,dx \ , \quad \int \frac{1}{\sqrt{a^2-x^2}}\,dx \ , \quad \int \frac{1}{(a^2-x^2)^{3/2}}\,dx$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx \qquad \int \frac{1}{x^2\sqrt{x^2-a^2}}\,dx$$

$$\int \frac{1}{x^2+a^2}\,dx$$

"inverse trig"

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x$$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}x$$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{4-\underbrace{(2\sin\theta)^2}_{4\sin^2\theta}}}\,d\theta = \int \frac{1}{\sqrt{4(\underbrace{1-\sin^2\theta}_{\cos^2\theta})}}\,d\theta = \int \frac{1}{\sqrt{4\cos^2\theta}}\,d\theta = \int \frac{1}{2\sqrt{\cos^2\theta}}$$

$$= \int \frac{1}{2|\cos\theta|}\,d\theta$$

$$(\sqrt{x^2} = |x|)$$

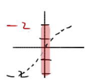
$$= \int \frac{1}{2\cos\theta}\,d\theta$$

b/c $\cos\theta > 0$ on $[\frac{-\pi}{2}, \frac{\pi}{2}]$

1. recognize form:

2. domain $\sqrt{4-x^2}$: $4-x^2<0$

$\Rightarrow x \in [-2, 2]$

3.  set: $x = 2\sin\theta$, with $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

3. $dx = 2\sin\theta\, d\theta$

$du = \sec\theta\tan\theta + \sec^2\theta$

$u = \sec\theta + \tan\theta$

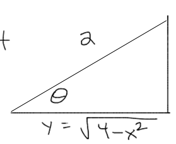
4. substitute / pythag

5. $$\int \frac{1}{2\cos\theta}\,d\theta = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\int \sec\theta \cdot \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \frac{1}{2}\int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \frac{1}{2}\int\frac{du}{u}$$

$$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

6. get x back.

create a Rt. Δ that matches $x = 2\sin\theta$



$\frac{x}{2} = \sin\theta = \frac{opp}{hyp}$

Pythagorean $\Rightarrow$ $2^2 = x^2 + y^2$

$4 - x^2 = y^2$

$\sqrt{4-x^2}$

7. use Δ to compute $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{adj}{hyp}} = \frac{hyp}{adj} = \frac{2}{\sqrt{4-x^2}}$ | $\tan\theta = \frac{opp}{adj} = \frac{x}{\sqrt{4-x^2}}$

8. Ans $= \frac{1}{2}\ln\left|\frac{2+x}{\sqrt{4-x^2}}\right| + C$

9. Check:

$$\frac{d}{dx}(\text{ans}) = \frac{1}{2}\cdot\frac{\dfrac{\frac{\sqrt{4-x^2}\sqrt{4-x^2}}{\sqrt{4-x^2}} + \frac{(2+x)\cdot x}{\sqrt{4-x^2}}}{(4-x^2)}}{\left(\frac{2+x}{\sqrt{4-x^2}}\right)} = \left(\frac{1}{2}\right)\frac{\dfrac{\overbrace{(4-x^2)+x^2+2x}^{4+2x}}{4-x^2}}{\dfrac{2+x}{\sqrt{4-x^2}}} = \frac{\dfrac{2+x}{4-x^2}}{\dfrac{2+x}{\sqrt{4-x^2}}} = \frac{\sqrt{4-x^2}}{4-x^2}$$

$$= \frac{1}{\sqrt{4-x^2}}$$

☺

$$\frac{\frac{A}{B}}{\frac{C}{B}} = \frac{A}{C}$$

$$= \frac{1}{2}\,\frac{\dfrac{x^2+2x+1}{\sqrt{4-x^2}}}{\dfrac{2+x}{\sqrt{4-x^2}}} = \frac{1}{2}\cdot\frac{(x+1)^2}{(x+2)}$$

Ex. Compute $\int \sqrt{9 - x^2}\, dx$ (related to area of circle w/ radius = 3)

① Recognize $x \in [-3, 3]$ $\Rightarrow$ set $x = 3\sin\theta$ $\theta \in [\frac{\pi}{2}, \frac{\pi}{2}]$

or $x = 3\cos\theta$ $\theta \in [0, \pi]$

② $dx = -3\sin\theta\, d\theta$

$$= -\int \sqrt{9 - (3\cos\theta)^2} \cdot 3\sin\theta\, d\theta = \int \sqrt{9 - 9\cos^2\theta} \cdot 3\sin\theta\, d\theta = \int \sqrt{9(\underbrace{1 - \cos^2\theta}_{\sin^2\theta})} \cdot 3\sin\theta\, d\theta$$

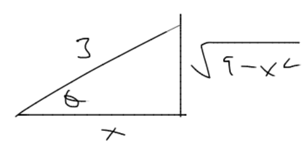
$$= -3\int \sqrt{(1 - \cos^2\theta)} \cdot 3\sin\theta\, d\theta$$

$$= 3\int |\sin\theta| \cdot \sin\theta\, d\theta$$

$$= -3\int \sin^2\theta\, d\theta = -3\int \frac{1 - \sin(2\theta)}{2}\, d\theta = -\frac{3}{2}\int 1 - \sin(2\theta)\, d\theta$$

$$= -\frac{3}{2}\left(\theta + \frac{\cos(2\theta)}{2}\right) \text{ get } x \text{ back}$$

$\cos\theta = \frac{x}{3}$



$$= \boxed{-\frac{3}{2}\left[\cos^{-1} x + \left(\frac{x}{3}\right)^2 - \left(\frac{\sqrt{9 - x^2}}{3}\right)^2\right]}$$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$