table integration & improper integrals

1. Use the end paper for this one - which formula applies?
a. table, formula # 117 with $u = e^x \to du = e^x\, dx$

$$\int \frac{e^x}{\sqrt{6e^x - e^{2x}}}\, dx = \int \frac{1}{\sqrt{2 \cdot 3 \cdot e^x - (e^x)^2}}(e^x)\, dx$$

$$\int \frac{1}{\sqrt{2au - u^2}}\, du = \cos^{-1}\left(\frac{a-u}{a}\right) + C$$

$$= \cos^{-1}\left(\frac{3-e^x}{3}\right) + C$$

b. table, formula # 113 with $u = e^x \to du = e^x\, dx$

$$\int e^x\sqrt{6e^x - e^{2x}}\, dx = \int \sqrt{2 \cdot 3 \cdot e^2 - (e^x)^2}\,(e^x)\, dx$$

$$= \int \sqrt{2au - u^2}\, du = \frac{u-a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

$$\frac{e^2-3}{2}\sqrt{6e^x - e^{2x}} + \frac{9}{2}\cos^{-1}\left(\frac{3-e^x}{3}\right) + C$$

2. Improper integrals

a.

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{b\to+\infty}\left[\int_1^b x^{-2}\,dx\right] = \lim_{b\to+\infty}\left[-x^{-1}\right]_1^b$$

$$= \lim_{b\to+\infty}\left[-\frac{1}{b}-\left(-\frac{1}{1}\right)\right] = 0+1=1$$

b.

$$\int_1^{+infty} \frac{1}{x}\,dx = \lim_{b\to+\infty}\left[\int_1^b \frac{1}{x}\,dx\right] = \lim_{b\to+\infty}[\ln|x|]_1^b$$

$$= \lim_{b\to+\infty}[\ln b - \ln 1] = \lim_{b\to+\infty}\ln b = +\infty$$

This integral diverges.

c.

$$\int_1^5 \frac{1}{\sqrt{x-1}}\,dx = \lim_{b\to1^+}\left[\int_b^5 \frac{1}{\sqrt{x-1}}\,dx\right] = \lim_{b\to1^+}\left[2\sqrt{x-1}\right]_b^5$$

$$= \lim_{b\to1^+}\left[2\sqrt{5-1}-2\sqrt{b-1}\right] = 2\sqrt{4}-0=4$$

d.

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_0^{+\infty} \frac{1}{x^2+1}\,dx$$

$$= \lim_{c\to-\infty}\int_c^0 \frac{1}{x^2+1}\,dx + \lim_{b\to+\infty}\int_0^b \frac{1}{x^2+1}\,dx$$

$$= \lim_{c\to-\infty}\left[\tan^{-1}x\right]_c^0 + \lim_{b\to+\infty}\left[\tan^{-1}x\right]_0^b$$

$$= \lim_{c\to-\infty}\left[\tan^{-1}0-\tan^{-1}c\right] + \lim_{b\to+\infty}\left[\tan^{-1}b-\tan^{-1}0\right]$$

$$= \left[0-\left(-\frac{\pi}{2}\right)\right] + \left[\frac{\pi}{2}-0\right] = \pi$$