Exam 1 - Chapter 7

Show all work to receive credit. Access to internet / graphing calculator / etc during the exam will result in a score of 0.

1. Integration by parts

$$\int x^4 e^{3x}\,dx = \frac{1}{2}x^4e^{2x} - x^3e^{2x} + \frac{3}{2}x^2e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{4}e^{2x} + C$$

sign	u		dv
$+$	x^4		e^{2x}
		$\searrow$	
$-$	$4x^3$		$\frac{1}{2}e^{2x}$
		$\searrow$	
$+$	$12x^2$		$\frac{1}{4}e^{2x}$
		$\searrow$	
$-$	$24x$		$\frac{1}{8}e^{2x}$
		$\searrow$	
$+$	24		$\frac{1}{32}e^{2x}$
		$\searrow$	
	0		$\frac{1}{16}e^{2x}$

2. Partial fractions

$$\int \frac{4x-1}{(x+2)(x-7)}\,dx \int \frac{1}{x+2} + \frac{3}{x-7}\,dx = \ln|x+2| + 3\ln|x-7| + C$$

Partial fractions decomposition:

$$\frac{4x-1}{(x+2)(x-7)} = \frac{A}{x+2} + \frac{B}{x-7} \Longrightarrow 4x-1 = A(x-7) + B(x+2)$$

$$\text{when } x = 7,\ \ 4(7) - 1 = A(7-7) + B(7+2) \ \Longrightarrow\ 27 = 9B \longrightarrow B = 3$$

$$\text{when } x = -2,\ \ 4(-2) - 1 = A(-2-7) + B(-2+2) \ \Longrightarrow\ -9 = -9A \longrightarrow A = 1$$

3. u-substitution, $u = x^5 \longrightarrow \frac{du}{dx} = 5x^4 \longrightarrow \frac{1}{5x^4}\,du = dx$

$$\int x^4\sec^2(x^5)\ dx = \int x^4\sec^2(u)\left(\frac{1}{5x^4}\right)\,du = \int \frac{1}{5}\sec^2 u\,du$$

$$= \frac{1}{5}\tan u + C = \frac{1}{5}\tan(x^5) + C$$

4. sine raised to odd power, so use $u = \cos\theta \to du = -\sin\theta\, d\theta$

$$\int \sin^5\theta\, d\theta = \int -(\sin^2\theta)(\sin^2\theta)(-\sin\theta)\, d\theta = \int -(1-\cos^2\theta)(1-cos^2\theta)(-\sin\theta)\, d\theta$$

$$= \int -(1-u^2)(1-u^2)\, du = \int -u^4 + 2u^2 - 1\, du = -\frac{1}{5}u^5 + \frac{2}{3}u^3 - u + C$$

$$= -\frac{1}{5}\cos^5\theta + \frac{2}{3}\cos^3\theta - \cos\theta + C$$

5.

Trig sub method: use $x = \frac{1}{2}\sin\theta \longrightarrow dx = \frac{1}{2}\cos\theta\, d\theta$ & $1 - 4x^2 = \cos^2\theta$
Triangle has $2x$ on the opposite leg, 1 on the hypotenuse, and $\sqrt{1-4x^2}$ on the adjacent leg.

$$\int \frac{x^5}{\sqrt{1-4x^2}}\, dx = \int \frac{\left(\frac{1}{2}\sin\theta\right)^5}{\sqrt{\cos^2\theta}} \cdot \frac{1}{2}\cos\theta\, d\theta = \int \frac{1}{64}\sin^5\theta\, d\theta$$

Use the answer to #4 above:

$$= \frac{1}{64}\left(-\frac{1}{5}\cos^5\theta + \frac{2}{3}\cos^3\theta - \cos\theta\right) + C$$

$$= -\frac{1}{320}(\sqrt{1-4x^2})^5 + \frac{1}{96}(\sqrt{1-4x^2})^3 - \frac{1}{64}\sqrt{1-4x^2} + C$$

U-subsitution method: $u = 1 - 4x^2 \to \frac{du}{dx} = -8x \to -\frac{1}{8x}\, du = dx$ & $x^2 = \frac{1-u}{4}$.

$$\int \frac{x^5}{\sqrt{1-4x^2}}\, dx = \int \frac{x^5}{\sqrt{u}}\left(-\frac{1}{8x}\right) du = \int -\frac{1}{8}x^4 u^{-1/2}\, du = \int -\frac{1}{8}\left(\frac{1-u}{4}\right)^2 u^{-1/2}\, du$$

$$= -\frac{1}{128}\int (1-u)^2 u^{-1/2}\, du = -\frac{1}{128}\int u^{3/2} - 2u^{1/2} + u^{-1/2}\, du$$

$$= -\frac{1}{128}\left(\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + 2u^{1/2}\right) + C$$

$$= -\frac{1}{320}(1-4x^2)^{5/2} + \frac{1}{96}(1-4x^2)^{3/2} - \frac{1}{64}\sqrt{1-4x^2} + C$$

6. (you need to do at least three of the following; additional ones may be done for extra credit.)
Evaluate at least three of the following integrals:

(a) Integration by parts, with $u = \sec^{-1} x \to du = \frac{1}{x\sqrt{x^2-1}}\,dx$ and $dv = 2x\,dx \to v = x^2$.

$$\int 2x \sec^{-1} x\,dx = x^2 \sec^{-1} x - \int \frac{x}{\sqrt{x^2-1}}\,dx$$
$$= x^2 \sec^{-1} x - \sqrt{x^2-1} + C$$

(b)

$$\int_0^{+\infty} e^{-4x}\,dx = \lim_{b\to+\infty}\left[\int_0^b e^{-4x}\,dx\right] = \lim_{b\to+\infty}\left[-\frac{1}{4}e^{-4x}\right]_0^b$$
$$= \lim_{b\to+\infty}\left[-\frac{1}{4}e^{-4b} - \left(-\frac{1}{4}e^0\right)\right] = \lim_{b\to+\infty}\left[-\frac{1}{4e^{4b}} + \frac{1}{4}\right] = \frac{1}{4}$$

(c) Integration by parts (once)

$$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int (\sec\theta\tan\theta)(\tan\theta)\ d\theta$$

$$\begin{array}{ll} u = \sec\theta & dv = \sec^2\theta\,d\theta \\ du = \sec\theta\tan\theta\,d\theta & v = \tan\theta \end{array}$$

$$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta$$
$$= \sec\theta\tan\theta - \int \sec\theta\left(\sec^2\theta - 1\right)\ d\theta = \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta$$

So . . .

$$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta$$

Then

$$2\int \sec^3\theta\,d\theta = \sec\theta\tan\theta + \int \sec\theta\,d\theta$$

Processing the last ingegral . . . . .

$$\int \sec\theta\,d\theta = \int \sec\theta\left(\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\right)\,d\theta$$
$$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\tan\theta+\sec\theta}\,d\theta = \ln|\tan\theta+\sec\theta| + C$$

Final answer

$$\Longrightarrow \int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

(d) Integration by parts (twice)

$$\int e^{2x} \sin 3x \, dx = -\frac{1}{3} e^{2x} \cos 3x - \int -\frac{2}{3} e^{2x} \cos 3x \, dx$$

$$\begin{array}{ll} u = e^{2x} & dv = \sin 3x \, dx \\ du = 2e^{2x} \, dx & v = -\frac{1}{3} \cos 3x \end{array}$$

$$-\frac{1}{3} e^{2x} \cos 3x + \int \frac{2}{3} e^{2x} \cos 3x \, dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \int \frac{4}{9} e^{2x} \sin 3x \, dx$$

$$\begin{array}{ll} u = \frac{2}{3} e^{2x} & dv = \cos 3x \, dx \\ du = \frac{4}{3} e^{2x} \, dx & v = \frac{1}{3} \sin 3x \end{array}$$

So . . .

$$\int e^{2x} \sin 3x \, dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} \int e^{2x} \sin 3x \, dx$$

$$\frac{13}{9} \int e^{2x} \sin 3x \, dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x$$

$$\int e^{2x} \sin 3x \, dx = -\frac{3}{13} e^{2x} \cos 3x + \frac{2}{13} e^{2x} \sin 3x + C$$

(e) Trig sub - use $x = 4 \sin \theta \longrightarrow dx = 4 \cos \theta \, d\theta$ & $16 - x^2 = 16 \cos^2 \theta$

$$\int \frac{x^2}{\sqrt{16 - x^2}} \, dx = \int \frac{(4 \sin \theta)^2}{\sqrt{16 \cos^2 \theta}} (4 \cos \theta) \, d\theta = \int 16 \sin^2 \theta \, d\theta = \int 8 - 8 \cos 2\theta \, d\theta$$

$$= 8\theta - 4 \sin 2\theta + C = 8\theta - 8 \sin \theta \cos \theta + C$$

$$= 8 \sin^{-1} \left(\frac{x}{4} \right) - 8 \left(\frac{x}{4} \right) \left(\frac{\sqrt{16 - x^2}}{4} \right) + C$$

(f) Partial fractions

$$\int \frac{3x^2 + 8x + 6}{(x+1)^3} \, dx = \int \frac{3}{x+1} + \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3} \, dx$$

$$= 3 \ln |x + 1| - \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C$$

Partial fractions decomposition

$$\frac{3x^2 + 8x + 6}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$3x^2 + 8x + 6 = A(x+1)^2 + B(x+1) + C = A(x^2 + 2x + 1) + B(x+1) + C$$

$$3x^2 + 8x + 6 = Ax^2 + (2A+B)x + A + B + C \Rightarrow \left\{ \begin{array}{rcll} A & = & 3 & \\ 2A + B & = & 8 & \longrightarrow B = 2 \\ A + B + C & = & 6 & \longrightarrow c = 1 \end{array} \right.$$