Improper Integrals: $\int_a^\infty f(x)\,dx \qquad \int_{-\infty}^a f(x)\,dx \qquad \int_{-\infty}^\infty f(x)\,dx$

$$\int_a^\infty f(x)\,dx \overset{def}{=} \lim_{R\to\infty} \int_a^R f(x)\,dx \overset{F.T.C.}{=} \lim_{R\to\infty} F(R) - F(a)$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

— computed as above —

Some examples ——

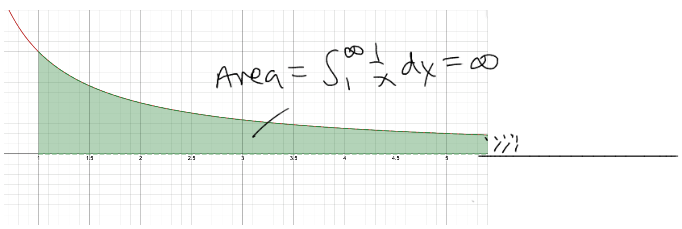
$$e^\infty = \lim_{R\to\infty} e^R$$

$$\int_1^\infty \frac{1}{x}\,dx = \ln|x|\Big|_1^\infty$$

both > 0 ⇒ drop |·| bars

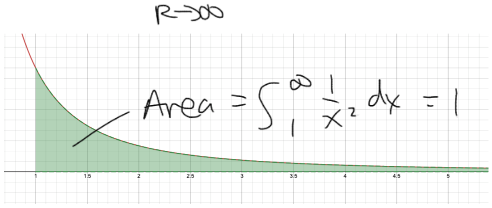
$y = \ln x$

$$= \lim_{R\to\infty} \ln(x)\Big|_1^R = \lim_{R\to\infty} \ln(R) - \ln(1) = \underbrace{\ln(\infty)}_{\text{means}} - \ln(1) = \infty$$



$$\int_1^\infty \frac{1}{x^2}\,dx = \int_1^\infty x^{-2}\,dx \overset{power}{=} \frac{x^{-1}}{-1} = -x^{-1}\Big|_1^\infty = \lim_{R\to\infty} -x^{-1}\Big|_1^R$$

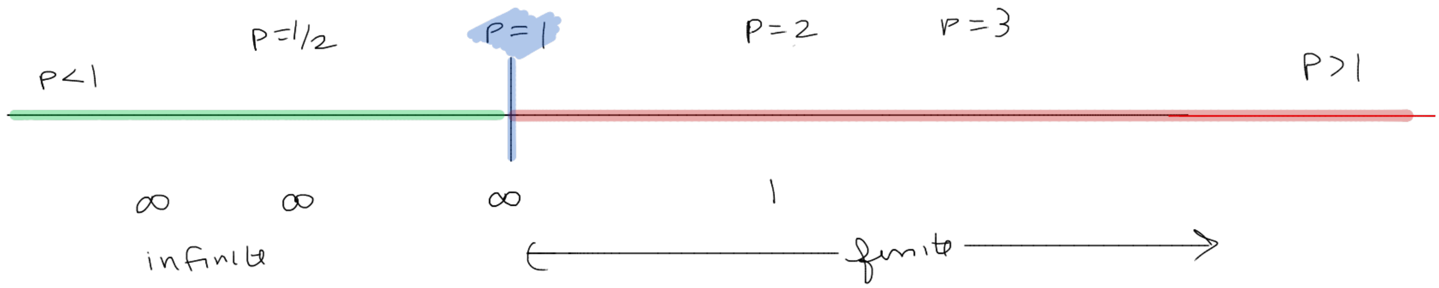
$$= \lim_{R\to\infty} -R^{-1} - (-1^{-1}) = \lim_{R\to\infty} \frac{-1}{R} - \frac{1}{-1} = \lim_{R\to\infty} \frac{-1}{R} + 1 = 1$$



$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \int_1^\infty \frac{1}{x^{1/2}}\,dx \overset{!}{=} \int_1^\infty x^{-1/2}\,dx = 2x^{1/2}\Big|_1^\infty = \lim_{R\to\infty} 2x^{1/2}\Big|_1^R$$

$$\lim_{R\to\infty} 2\sqrt{R} - 2\sqrt{1} = \infty$$

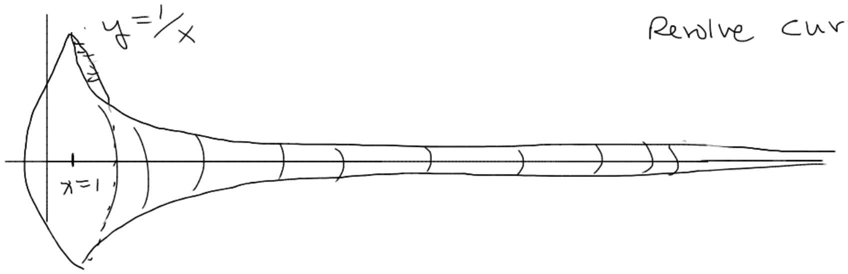
$\int \frac{1}{x^p}\,dx$ "p-integrals"



thm

$$\int \frac{1}{x^p}\,dx = \begin{cases} \text{finite} & p>1 \\ \infty & p \le 1 \end{cases}$$

Related to p-integrals: Painter's Paradox — Gabriel's Horn

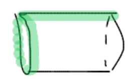


Revolve curve about x-axis

Vol: slice $\perp$ axis =  $r = 1/x$ $\quad A = \pi r^2 = \pi \frac{1}{x^2}$

$$V = \int \text{areaslice}\, dx = \int_1^\infty \pi \cdot \frac{1}{x^2}\, dx = \pi \qquad \text{(FINITE!)}$$

Suppose you fill Gabriel's Horn with paint. You only need a finite amount to fill it.

Now compute its surface area: recall: 

$$\int \text{circumference} \times \text{arc length}$$

Horn: $c = 2\pi r = 2\pi \cdot \frac{1}{x}$, arclength

$$\text{Surface Area} = \int_1^\infty 2\pi \cdot \frac{1}{x} \sqrt{1 + \frac{1}{x^2}}\, dx = \infty$$

But, you need an infinite amount to cover the outside.v

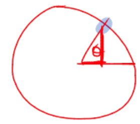
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \tan^{-1}x \Big|_{-\infty}^{\infty} = \tan^{-1}(\infty) - \tan^{-1}(-\infty) = \lim_{R\to\infty} \overset{\pi/2}{\tan^{-1}(R)} \overbrace{- \tan^{-1}(-R)}^{-(-\pi/2)} = \pi$$

$$= \int_{-\infty}^{a} \frac{1}{1+x^2}\,dx + \int_{a}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \underbrace{\tan^{-1}\Big|_{-\infty}^{a}} + \tan^{-1}\Big|_{a}^{\infty}$$

$$\tan^{-1}(a) - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - \tan^{-1}(a)$$

Recall:



$\tan\theta$ = slope det by θ

$$= \frac{\sin\theta}{\cos\theta} = \frac{\text{rise}}{\text{run}}$$

$\boxed{\tan^{-1}(x) = \text{angle that gives slope } x.}$