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sequences, ..., sequences, ...

1. Find the general term for each of the following sequences. Then determine whether or not the sequence converges. If the sequence does converge, find its limit.

(a)

$$\left\{\frac{1}{4^n}\right\}_{n=1}^{\infty} = \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}, \ldots$$

$$\lim_{n\to+\infty} \frac{1}{4^n} = 0 \Longrightarrow \text{ sequence converges to 0}$$

(b)

$$\left\{\frac{n}{n+2}\right\}_{n=1}^{\infty} = \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \ldots$$

$$\lim_{n\to+\infty} \frac{n}{n+2} = \lim_{n\to+\infty} 1 = 1 \Longrightarrow \text{ so sequence converges to 1}$$

(c)

$$\left\{\left(-\frac{1}{2}\right)^n\right\}_{n=0}^{\infty} = 1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$$

$$\lim_{n\to+\infty} \frac{(-1)^n}{2^n} = 0 \Longrightarrow \text{ so the sequence converges to 0}$$

(If you are having trouble with this limit - note that the top is bounded by 1 and -1 but the denominator is growing exponentially.)

(d)

$$\{(-1)^n\}_{n=0}^{\infty} = 1, -1, 1, -1, 1, \ldots$$

$$\lim_{n\to+\infty} (-1)^n \text{ does not exist } \Longrightarrow \text{ so the sequence diverges}$$

2. Write out the first six terms of the sequence below (decimal form). Does the sequence converge or diverge?

$$\left\{ n\sin\left(\frac{\pi}{n}\right)\right\}_1^{+\infty} = \sin\pi, 2\sin(\pi/2), 3\sin(\pi/3), 4\sin(\pi/4), 5\sin(\pi/5), 6\sin(\pi/6), \ldots$$

$$= 0, 2, 3\sqrt{3}/2, 2\sqrt{2}, \ldots$$

$$= 0, 2, 2.5981, 2.8284, 2.9389, 3, \ldots$$

$$\lim_{n\to+\infty} n\sin(\pi/n) = \lim_{n\to+\infty} \frac{\sin(\pi/n)}{1/n} = \lim_{n\to+\infty} \frac{-\frac{\pi}{n^2}\cos\left(\frac{\pi}{n}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to+\infty} \pi\cos\left(\frac{\pi}{n}\right) = \pi\cos 0 = \pi$$

The sequence converges to π.

3. Write the first five terms of the sequence defined below (decimal form). If the sequence converges, what is its limit?

$$a_{n+1} = \sqrt{a_n + 2}, \qquad a_1 = 0$$

$$\{a_n\} = 0, \sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \ldots$$

$$\simeq 0, 1.4142, 1.8478, 1.9616, 1.9904, \ldots$$

Assume that $\lim_{n\to+\infty} a_n = L$. So $\lim_{n\to+\infty} a_{n+1} = L$.

$$L = \lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} \sqrt{2+a_n} = \sqrt{2 + \lim_{n\to+\infty} a_n} = \sqrt{2+L}$$

Solve $L = \sqrt{2+L}$ for $L \Longrightarrow L = 2$.