## Comparison Tests

### (Direct) Comparison Test

Assumptions Matter!

For this comparison test to work need: $0 \le a_n \le b_n$
(two series, both positive, one "bigger than" the other.)

① If $\sum_{n=0}^{\infty} a_n$ diverges then $\sum_{n=0}^{\infty} b_n$ diverges

② If $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty}$ converges

Ex 1st use Integral Test to evaluate $\sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic), then use a comparison test to evaluate $\sum_{n=2}^{\infty} \frac{1}{3n-1}$.

① $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

$\int_1^{\infty} \frac{1}{x}\,dx = \ln|x|\Big|_1^{\infty} = \infty \Rightarrow$ series diverges. (albeit slowly)

Know $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

② $\sum_{n=2}^{\infty} \frac{1}{3n-1} = \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \dots$

Try to match assumptions of Comp Test: Check $\frac{1}{n} < \frac{1}{3n-1}$ ?

equiv. $3n-1 < n$, False

Comparison Test Doesn't Apply.

In order to apply comparison test need $b_n > \frac{1}{n}$ eg, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

( compare to $\frac{1}{n}$ ... $\frac{1}{n} \le \frac{1}{\sqrt{n}+1}$ $\forall\ n \ge 1$ ↳ some finite #

$\sqrt{n}+1 \le n$

$\sqrt{n} \le n-1$

$n \le n^2 - 2n + 1$

$0 \le n^2 - 3n + 1$

$\sqrt{2}+1 \le 2$

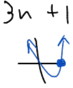
$\sqrt{4}+1 \le 4$

$n=1$ ✗

$n=2$ ✗

$n=3$ ✓

$n=4$

 $\infty$ # of "n" satisfying this ineq,

$\Rightarrow$ Comparison Test applies:

① Know $\frac{1}{n} \le \frac{1}{\sqrt{n}+1}$ (from above)

② $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges So Comp Test $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges

MA163 FRI

Review + Integral & Comparison Tests

Series: $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} a_n}_{S_N}$

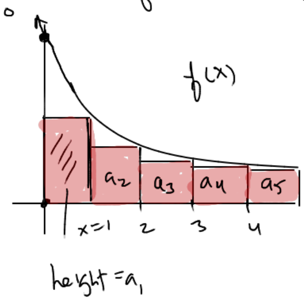
Geometric series: $\sum_{n=0}^{\infty} c r^n$ $c = 1^{st}$ term, $r =$ common ratio $= \begin{cases} \frac{c}{1-r} & |r|<1 \\ \text{DNE} & |x| \geq 1 \end{cases}$

Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges

Integral Test: (only applies to positive series!)

For $\sum_{n=0}^{\infty} a_n$ if $a_n = f(n)$ where $f(x)$ is positive continuous decreasing

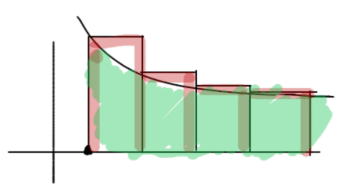
then




curve dominates (above) rectangle.

If the area under the curve (from 0 to infinity) is finite, then the series converges

$\boxed{\text{If } \int_0^{\infty} f(x)\,dx \text{ converges then } \sum_{n=0}^{\infty} a_n \text{ converges.}}$



Here rectangles are above curve

$\boxed{\int_1^{\infty} f(x)\,dx \text{ diverges then } \sum_{n=0}^{\infty} a_n \text{ diverges}}$

Ex $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$

$\frac{1}{\sqrt{n+3}} = f(n)$ where $f(x) = \frac{1}{\sqrt{x+3}}$

$\int_1^{\infty} \frac{1}{\sqrt{x+3}}\,dx \overset{\substack{u=x+3 \\ du=dx \\ x=1 \Rightarrow u=4}}{=\!=\!=} \int_4^{\infty} \frac{1}{\sqrt{u}}\,du = \int_4^{\infty} u^{-1/2}\,du$

expect this series to diverge

$= 2u^{1/2}\Big|_4^{\infty}$

$2(\infty)^{1/2} - 2(4)^{1/2}$

$\infty - 4 = \infty$

Int. Test $\Rightarrow$ series diverges

Note: $\int_0^{\infty} \frac{1}{\sqrt{x+3}}\,dx = \infty$ would not satisfy assumptions of int test

Ex. $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

Harmonic

Apply Int. Test $\frac{1}{n} = f(n)$ where $f(x) = \frac{1}{x}$ $\left\{\begin{matrix}\text{make sure} \\ \text{assumptions} \\ \text{are true}\end{matrix}\right\}$

1. continuous ✓
2. positive on $(1,\infty)$ ✓
3. decreasing: $\left(\frac{1}{x}\right)' = -\frac{1}{x^2} < 0$

$\int_1^{\infty} \frac{1}{x}\,dx = \ln|x|\Big|_1^{\infty} = \ln(\infty) - \ln(1) = \infty - 0 = \infty$

$\Rightarrow$ series diverges!

## Comparison Tests

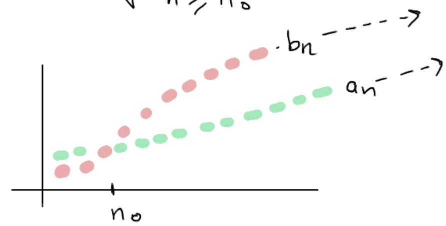
(Only applies to positive series)

given two series –

$$\sum_{n=0}^{\infty} a_n \qquad \sum_{n=0}^{\infty} b_n \qquad 0 \le a_n \le b_n$$

for all (eventual) $n$

$\forall\, n \geq n_0$




If $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges

If $\sum_{n=0}^{\infty} a_n$ diverges then $\sum_{n=0}^{\infty} b_n$ diverges

Ex. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+3}$

try to apply comparison test: $\frac{1}{n} > \frac{1}{n+3}$ For this comparison to work, need $\sum \frac{1}{n}$ converge, false

so this comparison doesn't work

Int: $\frac{1}{x+3} = f(x)$ cts pos dec $\left(\frac{1}{x+3}\right)' = -(x+3)^{-2} = \frac{-1}{(x+3)^2}$ 🙂

$\int_1^{\infty} \frac{1}{x+3}\,dx = \ln|x+3| = \infty \Rightarrow$ series diverge

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}+3}$

$\boxed{\text{since } \tfrac{1}{3}\textstyle\sum \frac{1}{n} \text{ diverges, } \sum \frac{1}{\sqrt{n}+3} \text{ diverges}}$

? $\frac{1}{n} < \frac{1}{\sqrt{n}+3} \iff$ TRUE b/c $\longrightarrow$

$\Sigma \downarrow \infty \qquad \Sigma \downarrow \infty$

$\sqrt{n}+3 < n$

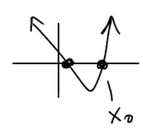
$\sqrt{n} < n-3$

$n < (n-3)^2$

$n < n^2 - 6n + 9$

$0 < n^2 - 7n + 9$

$\updownarrow$ imagine $0 < x^2 - 7x + 9$




eventually this holds

i.e, true $\forall\, x > x_0$