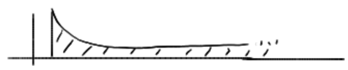
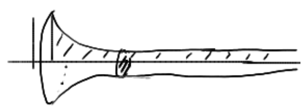
Wed • Week 5

Last time: Painter's Paradox & Gabriel's Horn



infinite area $\int_1^\infty \frac{1}{x} = \lim \ln x \Big|_1^\infty = \infty$ the area accumulates as $\frac{1}{x}$



finite volume $\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi$ B/c of πr^2, the area accumulates as $\pi\left(\frac{1}{x}\right)^2$ or $\pi\left(\frac{1}{x^2}\right)$

To compare $\frac{1}{x}$ & $\frac{1}{x^2}$ look at sequences

$\frac{1}{x}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$

$\frac{1}{x^2}$: $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \dots$

& later at series

$\sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{N}$ | $\sum_{n=1}^{N} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots + \frac{1}{N^2}$

Sequences:

1. Find formula for general term
2. Calculate limit if it exists
3. Determine convergence / divergence

Def: An infinite sequence is ordered list of numbers:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

a term — the index variable

shorthand: $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$

Ex: Suppose $\{a_n\} = 2, 4, 6, 8, \ldots$ what is a_j? $\longrightarrow$ j^2 (here we start w/ 1, sometimes 0.)

$\{b_n\} = 2, 4, 8, 16, 32 \longrightarrow b_j = 2^j$

Def'n: Recurrance Rel'n: One term given explicitly, the others are defined by previous terms.

Ex. $a_1 = 2, \quad a_n = (a_{n-1})^2 \Rightarrow$

$a_2 = a_1^2 = 4$

$a_3 = 4^2 = 16$

$a_4 = 16^2 =$

$2, 4, 16, 2^8, 2^{16}, \ldots$

$2^1, 2^2, 2^4, 2^8, 2^{16}, \ldots 2^{2^n}$

$\boxed{a_n = 2^{2^n}}$

Two Important Types of sequences

arithmetic sequence: difference b/w consecutive terms is constant.

$2, 7, 12, 17, 22, \ldots$ $\quad \begin{cases} a_1 = 2 \\ a_n = a_{n-1} + 5 \end{cases}$

$\Rightarrow a_2 = 2 + 5$

$a_3 = a_2 + 5 = 2 + 5 + 5 = 2 + 2(5)$

$a_4 = a_3 + 5 = 2 + 2(5) + 5 = 2 + 3(5)$

$\vdots$

$a_n = 2 + (n-1)5 = 2 + 5n - 5 = 5n - 3$

$\Rightarrow$ General Form of Arithmetic Seq: $\quad a_n = c \cdot n + b$

Geometric Seq: Ratio of consecutive terms is constant.

Ex: $\frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{36}, \frac{1}{72}, \ldots$ $\quad \frac{\frac{1}{6}}{\frac{1}{12}} = 2$

$\left.\begin{array}{l} a_1 = 1/6 \\ a_n = \frac{1}{2} a_{n-1} \end{array}\right\}$

$a_2 = \frac{1}{2} \cdot \frac{1}{6}$

$a_3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \left(\frac{1}{2}\right)^2 \frac{1}{6}$

$a_4 = \left(\frac{1}{2}\right)^3 \cdot \frac{1}{6}$

$\vdots$

$a_n = \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{6}$

$= \left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{2}\right)^{-1} \cdot \frac{1}{6}$

$= \left(\frac{1}{2}\right)^n \cdot \frac{2}{6}$

$= \left(\frac{1}{2}\right)^n \frac{1}{3}$

$= \underbrace{\frac{1}{3}}_{a} \big(\underbrace{\tfrac{1}{2}}_{r}\big)^n$

Ex: $2, \frac{-2}{3}, \frac{2}{9}, \frac{-2}{27}, \frac{2}{81}, \ldots$ $\quad \frac{\frac{2}{9}}{\frac{-2}{3}} = -\frac{1}{3}$, same

$$\boxed{\text{Gen'l Form of Geom. Sequence} \qquad a_n = a r^n}$$

Exercise: (a) $\left\{-\frac{1}{2}, \frac{2}{3}, \frac{-3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}, \frac{-7}{8}, \ldots\right\}$ $\quad a_n = ?$ $\ldots = (-1)^n\left(\frac{n}{n+1}\right)$

(b) $\left\{\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \ldots\right\}$ $\quad a_n = ?$ $\ldots = \frac{3^n}{3n+1}$

(c) $\left\{\underset{n=1}{\frac{1}{5}}, \underset{n=2}{\frac{-1}{7}}, \underset{n=3}{\frac{1}{9}}, \underset{n=4}{\frac{-1}{11}}, \ldots\right\}$ $\quad = \frac{(-1)^{n+1}}{2n+3}$ $\quad 5 + 2(n-1)$

(d) $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ $\quad a_n = a_{n-1} + a_{n-2}$ $\quad$ Fibanacci.

Limit of a Sequence

(a) $\{1 + 3n\} = a_n$, $\lim_{n\to\infty} a_n = \infty$

(b) $\left\{1 - \left(\frac{1}{2}\right)^n\right\} = b_n$, $\lim_{n\to\infty} b_n = 1$

(c) $\{(-1)^n\} = c_n$, $\lim_{n\to\infty} c_n =$ undefined

(d) $\left\{\frac{(-1)^n}{n}\right\} = d_n$, $\lim_{n\to\infty} d_n = 0$

Def'n: Convergent & Divergent

Given a seq. $\{a_n\}$ if the terms a_n become arbitraily close to a finite number L as n becomes suff. large we say the sequence converges to L.

$$\lim_{n \to \infty} a_n = L$$

If $\{a_n\}$ is not convergent it is a divergent seq.

Note! For $x \in \mathbb{R}$, $n \in \mathbb{N}$ $\quad \lim_{x \to \infty} f(x) = \lim_{n \to \infty} f(n)$

Thm: Limits of Geometric Sequences

$r^n \longrightarrow 0$ if $0 < r < 1$

$r^n \longrightarrow 1$ if $r = 1$

$r^n \longrightarrow \infty$ if $r > 1$

If $0 < r < 1$ then $r = \frac{1}{x}$ for some $x > 1$. then

$$\lim_{n \to \infty} r^n = \lim_{n \to \infty} \left(\frac{1}{x}\right)^n = \lim_{n \to \infty} \frac{1}{x^n} = 0.$$

Recall: Alg. Limit Laws