world of series . . . .

1. Determine whether or not the series converges. If the series converges, then find its sum.

(a) Geometric series, $r = 2/3$, convergent

$$\sum_{k=0}^{\infty} 2\left(\frac{2}{3}\right)^k = 2 + \frac{4}{3} + \frac{8}{9} + \cdots = \frac{2}{1-\frac{2}{3}} = 6$$

(b) Geometric series, $r = 2/3$, convergent

$$\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k+2} = \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots = \frac{\frac{8}{27}}{1-\frac{2}{3}} = \frac{8}{9}$$

(c) Geometric series, $r = -3/2$, divergent

$$\sum_{k=1}^{\infty} \left(-\frac{3}{2}\right)^k = -\frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$$

(d) Geometric series, $r = 125/7$, divergent

$$\sum_{k=0}^{\infty} 5^{3k}7^{1-k} = 7 + 125 + \frac{15625}{7} + \cdots$$

2. In each part, find all values of c for which the series converges, and find the sum of the series (the sum will still have a "c" in it).

(a) Geometric series, $r = -c^2$.

$$c - c^3 + c^5 - c^7 + c^9 - \cdots = \frac{c}{1+c^2} \text{ for } -1 < c < 1, \text{ otherwise diverges}$$

(b) Geomeric series, $r = e^{-c}$

$$e^{-c} + e^{-2c} + e^{-3c} + e^{-4c} + e^{-5c} + \cdots = \frac{e^{-c}}{1-e^{-x}} \text{ for } c > 0, \text{ otherwise diverges}$$

3. Show that

$$\sum_{k=1}^{\infty} \frac{2}{k^2+2k} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2} \right) = \frac{3}{2}$$

Just add the two fractions $1/k$ and $-1/(k+2)$ to get $2/(k^2+2k)$ - this demonstrates the first equality.

Based on the second version of the series,

$$a_1 = 1 - \frac{1}{3}, a_2 = \frac{1}{2} - \frac{1}{4}, a_3 = \frac{1}{3} - \frac{1}{5}, a_4 = \frac{1}{4} - \frac{1}{6}, \ldots$$

$$s_1 = a_1 = 1 - \frac{1}{3}$$

$$s_2 = s_1 + a_2 = 1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$$

$$s_3 = s_2 + a_3 = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$$

$$s_4 = s_3 + a_4 = 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$$

$$\Longrightarrow s_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

Then

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2} \right) = \lim_{n \to +\infty} s_n = \lim_{n \to +\infty} \left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right] = \frac{3}{2}$$