Exam 1 - Chapter 7

Show all work to receive credit. Access to internet / graphing calculator / etc during the exam will result in a score of 0.

1. Integration by parts

$$\int x^4 \cos 2x \, dx = \frac{1}{2}x^4 \sin 2x + x^3 \cos 2x - \frac{3}{2}x^2 \sin 2x - \frac{3}{2}x \cos 2x + \frac{3}{4} \sin 2x + C$$

sign	u		dv
$+$	x^4		$\cos 2x$
		$\searrow$	
$-$	$4x^3$		$\frac{1}{2}\sin 2x$
		$\searrow$	
$+$	$12x^2$		$-\frac{1}{4}\cos 2x$
		$\searrow$	
$-$	$24x$		$-\frac{1}{8}\sin 2x$
		$\searrow$	
$+$	24		$\frac{1}{16}\cos 2x$
		$\searrow$	
	0		$\frac{1}{32}\sin 2x$

2. Partial fractions

$$\int \frac{2x+1}{(x-2)(x-7)} \, dx = \int \frac{-1}{x-2} + \frac{3}{x-7} \, dx = -\ln|x-2| + 3\ln|x-7| + C$$

Partial fractions decomposition:

$$\frac{2x+1}{(x-2)(x-7)} = \frac{A}{x-2} + \frac{B}{x-7} \Longrightarrow 2x+1 = A(x-7) + B(x-2)$$

$$\text{when } x = 7,\ \ 2(7)+1 = A(7-7) + B(7-2) \ \ \Longrightarrow \ \ 15 = 5B \longrightarrow B = 3$$

$$\text{when } x = 2,\ \ 2(2)+1 = A(2-7) + B(2-2) \ \ \Longrightarrow \ \ 5 = -5A \longrightarrow A = -1$$

3. u-substitution, $u = x^5 \longrightarrow \frac{du}{dx} = 5x^4 \longrightarrow \frac{1}{5x^4}\, du = dx$

$$\int x^4 \sec^2(x^5) \ dx = \int x^4 \sec^2(u) \left(\frac{1}{5x^4}\right) \ du = \int \frac{1}{5} \sec^2 u \, du$$

$$= \frac{1}{5} \tan u + C = \frac{1}{5} \tan(x^5) + C$$

4. tangent raised to odd power, so use $u = \sec\theta \to du = \sec\theta\tan\theta\, d\theta$

$$\int \tan^3\theta\sec^3\theta\, d\theta = \int \sec^2\theta\tan^2\theta(\sec\theta\tan\theta)\, d\theta = \int \sec^2\theta(\sec^2\theta - 1)(\sec\theta\tan\theta)\, d\theta$$

$$= \int u^2(u^2-1)\, du = \int u^4 - u^2\, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta + C$$

5.

Trig sub method: use $x = 2\tan\theta \longrightarrow dx = 2\sec^2\theta\, d\theta$ & $x^2+4 = 4\sec^2\theta$
Triangle has x on the opposite leg, 2 on the adjacent leg, and $\sqrt{x^2+4}$ on the hypotenuse.

$$\int x^3\sqrt{x^2+4}\, dx = \int (2\tan\theta)^3\sqrt{4\sec^2\theta}(2\sec^2\theta)\, d\theta = \int 32\tan^3\theta\sec^3\theta\, d\theta$$

Use the answer to #4 above:

$$= 32\left(\frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta\right) + C = 32\left(\frac{1}{5}\left(\frac{\sqrt{x^2+4}}{2}\right)^5 - \frac{1}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3\right) + C$$

$$= \frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C$$

U-subsitution method: $u = x^2+4 \to \frac{du}{dx} = 2x \to \frac{1}{2x}\, du = dx$ & $x^2 = u-4$.

$$\int x^3\sqrt{x^2+4}\, dx = \int x^3\sqrt{u}\left(\frac{1}{2x}\right)\, du = \int \frac{1}{2}x^2\sqrt{u}\, du = \int \frac{1}{2}(u-4)u^{1/2}\, du$$

$$= \int \frac{1}{2}u^{3/2} - 2u^{1/2}\, du = \frac{1}{2}\cdot\frac{2}{5}u^{5/2} - 2\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C$$

6. (you need to do at least three of the following; additional ones may be done for extra credit.)
Evaluate at least three of the following integrals:

(a) Integration by parts, with $u = \sec^{-1} x \to du = \frac{1}{x\sqrt{x^2-1}}\,dx$ and $dv = 2x\,dx \to v = x^2$.

$$\int 2x \sec^{-1} x\,dx = x^2 \sec^{-1} x - \int \frac{x}{\sqrt{x^2-1}}\,dx$$
$$= x^2 \sec^{-1} x - \sqrt{x^2-1} + C$$

(b) improper integral - break into two pieces at the "bad" point 0

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{2} \frac{1}{x^2}\,dx$$

Check for convergence / divergence of the second integral

$$\int_0^2 \frac{1}{x^2}\,dx = \lim_{b\to 0^+} \left[\int_0^2 \frac{1}{x^2}\,dx\right] = \lim_{b\to 0^+} \left[-\frac{1}{x}\Big|_b^2\right] = \lim_{b\to 0^+} \left[-\frac{1}{2} - \left(-\frac{1}{b}\right)\right] = +\infty$$

Answer: The integral diverges.

(c) Trig sub- let $x = 3\sec\theta \to dx = 3\sec\theta\tan\theta\,d\theta$ & $x^2 - 9 = 9\tan^2\theta$
Triangle has label x on the hypotenuse, 3 on the adjacent leg, and $\sqrt{x^2-9}$ on the opposite leg.

$$\int \frac{\sqrt{x^2-9}}{x^4}\,dx = \int \frac{\sqrt{9\tan^2\theta}}{(3\sec\theta)^4} \cdot 3\sec\theta\tan\theta\,d\theta = \int \frac{9\tan^2\theta\sec\theta}{81\sec^4\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{9}\int \left(\frac{\sin\theta}{\cos\theta}\right)^2 \left(\frac{\cos\theta}{1}\right)^3 d\theta = \frac{1}{9}\int \sin^2\theta\cos\theta\,d\theta$$

$$= \frac{1}{27}\sin^3\theta + C = \frac{1}{27}\left(\frac{\sqrt{x^2-9}}{x}\right)^3 + C = \frac{(x^2-9)^{3/2}}{27x^3} + C$$

(d) Integration by parts (twice)

$$\int e^{3x}\sin 4x\,dx = -\frac{1}{4}e^{3x}\cos 4x - \int -\frac{3}{4}e^{3x}\cos 4x\,dx$$

$$\begin{array}{cc} u = e^{3x} & dv = \sin 4x\,dx \\ du = 3e^{3x}\,dx & v = -\frac{1}{4}\cos 4x \end{array}$$

$$-\frac{1}{4}e^{3x}\cos 4x + \int \frac{3}{4}e^{3x}\cos 4x\,dx = -\frac{1}{4}e^{3x}\cos 4x + \frac{3}{16}e^{3x}\sin 4x - \int \frac{9}{16}e^{3x}\sin 4x\,dx$$

$$\begin{array}{cc} u = \frac{3}{4}e^{3x} & dv = \cos 4x\,dx \\ du = \frac{9}{4}e^{3x}\,dx & v = \frac{1}{4}\sin 4x \end{array}$$

So . . .

$$\int e^{3x}\sin 4x\,dx = -\frac{1}{4}e^{3x}\cos 4x + \frac{3}{16}e^{3x}\sin 4x - \frac{9}{16}\int e^{3x}\sin 4x\,dx$$

$$\frac{25}{16}\int e^{3x}\sin 4x\,dx = -\frac{1}{4}e^{3x}\cos 4x + \frac{3}{16}e^{3x}\sin 4x$$

$$\int e^{3x}\sin 4x\,dx = -\frac{4}{25}e^{3x}\cos 4x + \frac{3}{25}e^{3x}\sin 4x + C$$

(e) Trig sub - use $x = 4\sin\theta \longrightarrow dx = 4\cos\theta\,d\theta$ & $16 - x^2 = 16\cos^2\theta$

$$\int \frac{x^2}{\sqrt{16-x^2}}\,dx = \int \frac{(4\sin\theta)^2}{\sqrt{16\cos^2\theta}}(4\cos\theta)\,d\theta = \int 16\sin^2\theta\,d\theta = \int 8 - 8\cos 2\theta\,d\theta$$

$$= 8\theta - 4\sin 2\theta + C = 8\theta - 8\sin\theta\cos\theta + C$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - 8\left(\frac{x}{4}\right)\left(\frac{\sqrt{16-x^2}}{4}\right) + C$$

(f) Partial fractions

$$\int \frac{x^2+4x+6}{(x+1)^3}\,dx = \int \frac{1}{x+1} + \frac{2}{(x+1)^2} + \frac{3}{(x+1)^3}\,dx$$

$$= \ln|x+1| - \frac{2}{x+1} - \frac{3}{2(x+1)^2} + C$$

Partial fractions decomposition

$$\frac{x^2+4x+6}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$x^2+4x+6 = A(x+1)^2 + B(x+1) + C = A(x^2+2x+1) + B(x+1) + C$$

$$x^2+4x+6 = Ax^2 + (2A+B)x + A + B + C \Rightarrow \left\{\begin{array}{rcll} A & = & 1 & \\ 2A+B & = & 4 & \longrightarrow B = 2 \\ A+B+C & = & 6 & \longrightarrow c = 3 \end{array}\right.$$