MA163 wk 6

Limit Comparison Test

For $\{a_n\}$, $\{b_n\}$ positive, w/ $L = \lim_{n\to\infty} \frac{a_n}{b_n}$

- If $L > 0$ (limit exists) $\neq 0$

$\frac{\lim a_n}{\lim b_n} = L \underset{\text{mult}}{\overset{\text{cross}}{\Rightarrow}} \lim a_n = L \cdot \lim b_n$ (L: some #)

the terms a_n, b_n are growing @ 'roughly' the same rate

$\sum a_n$ converges $\Leftrightarrow$ $\sum b_n$ converges

($\Leftrightarrow$: if and only if)

- If $L = \infty$ $\boxed{\frac{\lim a_n}{\lim b_n} = \infty}$ $\Rightarrow$ $\underbrace{\lim a_n}_{=0} = \infty \cdot \lim b_n$ $\Rightarrow$ $\boxed{\sum b_n \text{ converges}}$

only way this is possible is if $\lim b_n = 0$

If $\sum a_n$ converges (i.e., $\lim_{n\to\infty} a_n = 0$)

the b_n terms go to zero FASTER than the terms of a convergent series, so the b-series has to converge

- If $L = 0$ $\frac{\lim a_n}{\lim b_n} = 0$ $\Rightarrow$ $\boxed{\text{If } \sum b_n \text{ converge then } \sum a_n \text{ converges}}$

i..e., the denom is going to infinity much faster than the numerator

Ex $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 5n + 7}$. Does the series converge?

think:
1. fraction
2. focus on the first terms: $\sim \frac{n^2}{n^4} = \frac{1}{n^2}$ ... $\sum_{n=2}^{\infty} \frac{1}{n^2}$ ... p-series $p = 2 > 1$ } converge $\Rightarrow$ compare given to $\frac{1}{n^2}$

3. $\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{n^2}{n^4 - 5n + 7}} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n^4 - 5n + 7}{n^2} = \lim_{n\to\infty} \frac{n^4 - 5n + 7}{n^4} = \frac{\infty}{\infty}$

L'H $\frac{4n^3 - 5}{4n^3} = \frac{\infty}{\infty}$, L'H $\frac{12n^2}{12n^2} = 1$

4. By L.C.T. since $\sum \frac{1}{n^2}$ converges our series converges.

Ex $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

$\sum \frac{1}{n}$ diverges
HARMONIC
int. test $\int_1^{\infty} \frac{1}{x}\,dx = \infty$

1. see: fraction, focus on leading terms, see 1/n which you know diverges (harmonic)

2. now we have 1/n a diverging series to compare:

$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2+1}}} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{\sqrt{n^2+1}}{1} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n} \approx \frac{\sqrt{n^2}}{n} = 1.$ $\Rightarrow \sum \frac{1}{n} \sim \sum \frac{1}{\sqrt{n^2+1}}$

($\sum \frac{1}{n}$: div.)

$\Rightarrow$ L.C.T. $\Rightarrow$ this diverges

How to deal w/ series that are not always positive:

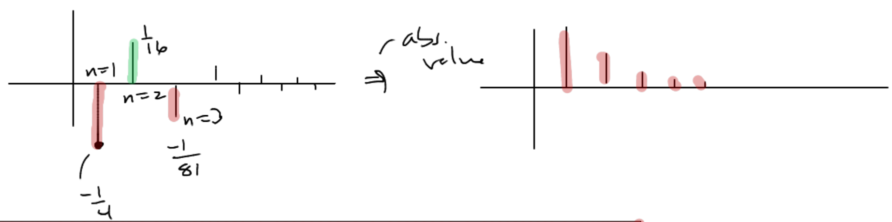
<u>Def</u>: Absolute Converge

the series $\sum a_n$ converge absolutely if $\sum |a_n|$ converges.

<u>Ex</u>

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ converges absolutely b/c $\sum \left| \frac{(-1)^n}{n^4} \right| = \sum \frac{1}{n^4} \rightarrow$ converges

(this looks like



$$\boxed{\text{Theorem: Absolute Convergence implies convergence. If } \sum |a_n| \text{ converges then } \sum a_n \text{ converge}}$$

taking absolute values makes it much harder to converge. if the abs values series converges well, then the one that is sometimes +/- does converge``

<u>Ex</u> $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2}$

does this converge?

Yes. this series converge absolutely —

$\sum \left| \frac{(-1)^n}{n^2} \right| = \sum \frac{1}{n^2} =$ p-series w/ $n>1 \Rightarrow$ converges

And if a series conv. absolutely then it <u>converges</u>