# Algebra Tricks to evaluate series

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)} = \frac{2}{1(2)} + \frac{2}{2\cdot 3} + \frac{2}{3\cdot 4} + \dots = 1 + \frac{1}{3} + \frac{1}{6} + \dots$$

Hint:

$$\int \frac{2}{n(n+1)}\,dx$$

Partial Fraction

↑ not geometric

$$\frac{2}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$2 = A(n+1) + Bn = (A+B)n + A$$

$$\left.\begin{array}{l} A+B=0 \\ A=2 \end{array}\right\} \quad B=-2$$

$$\sum_{n=1}^{\infty} \frac{2}{n} - \frac{2}{n+1}$$

$$S_1 = 2 - \frac{2}{2} = 1 \quad (n=1)$$

$$S_2 = 1 + \underbrace{\frac{2}{2} - \frac{2}{3}}_{n=2} = 2 - \frac{2}{3}$$

$$S_3 = 2 - \frac{2}{3} + \underbrace{\frac{2}{3} - \frac{2}{4}}_{n=3} = 2 - \frac{2}{4}$$

$$S_4 = \underbrace{2 - \frac{2}{4}}_{S_3} + \underbrace{\frac{2}{4} - \frac{2}{5}}_{n=4} = 2 - \frac{2}{5}$$

$$S_n = 2 - \frac{2}{n+1}$$

the limit of this IS the infinite sum

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - \frac{2}{n+1}$$

$$= 2$$

warmup:

$$\sum_{n=0}^{\infty} 3\left(\frac{3}{4}\right)^n$$

geometric,
$a=3$, $r=\frac{3}{4}$

$$= \frac{3}{1-\frac{3}{4}} = \frac{3}{(1/4)}$$

$$= 12$$

what you know:

- this is a <u>series</u>
- the terms are:

$$\underset{n=0}{3}, \quad \underset{n=1}{3\left(\tfrac{3}{4}\right)}, \quad \underset{n=2}{3\left(\tfrac{3}{4}\right)^2}, \ldots$$

see any pattern?

ratio of consecutive terms $= \frac{3}{4}$

- Series whose ratios of consecutive terms are <u>geometric</u>.

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}$$

- converges when $-1 < r < 1$

Recall –

$$\begin{array}{rl} a + ar + ar^2 + ar^3 + \ldots &= S \\ -\left(ar + ar^2 + ar^3 + ar^4 + \ldots \right. &= \left. rS\right) \\ \hline a &= S - rS = S(1-r) \end{array}$$

$$\boxed{\frac{a}{1-r} = S}$$

Next

Idea: Realize this is a geom. series
1. re-index s.t. we start @ 0.

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k+2} = \left(\frac{3}{4}\right)^{2} \cdot \left(\frac{3}{4}\right)^{k}$$

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{2} \left(\frac{3}{4}\right)^{n+1}$$

$$\|$$

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{n+3}$$

Terms

$\left(\frac{3}{4}\right)^{3}$, $\left(\frac{3}{4}\right)^{4}$

$k=1$ $\quad$ $k=2$

$n=0$

$$\sum_{n=0}^{\infty} ar^{n}$$

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$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k+2} \quad - \quad \frac{\left(\frac{3}{4}\right)^{3}}{k=1}$$

$$\boxed{\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^{k+3}} \quad - \quad \frac{\left(\frac{3}{4}\right)^{3}}{k=0}$$

$$A^{B+C} = A^{B} \cdot A^{C}$$

$$\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^{3} \cdot \left(\frac{3}{4}\right)^{k} = $$

geometric

$$\sum_{k=0}^{\infty} ar^{n}$$

$$\frac{a}{1-r} = \frac{\frac{27}{64}}{1-\frac{3}{4}} = \frac{27}{64} \cdot \frac{4}{1} = \boxed{\frac{27}{16}}$$

#3 evaluate

$$\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n = 5 - 5\left(\frac{4}{3}\right) + 5\left(\frac{16}{9}\right) - 5\left(\frac{64}{27}\right) + 5\left(\frac{256}{81}\right) - \ldots$$

geometric: $\boxed{\sum_{n=0}^{\infty} ar^n} = \frac{a}{1-r}$ converge precisely when $-1 < r < 1$

diverges

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#4

$$\sum_{n=0}^{\infty} 4^{2n} \cdot 5^{(1-3n)} = \sum_{n=0} 5\, 4^{2n} \cdot 5^{-3n}$$

$$= \sum_{n=0}^{\infty} 5 \cdot (4^2)^n \cdot (5^{-3})^n$$

$$= \sum_{n=0}^{\infty} 5 \left(16 \cdot 5^{-3}\right)^n$$

$$= \boxed{\sum_{n=0}^{\infty} 5\left(\frac{16}{125}\right)^n} = \frac{5}{1-\left(\frac{16}{125}\right)} = \frac{5}{\frac{109}{125}} = \frac{625}{109}$$

$\approx 6$

- $A^n \cdot B^n = (AB)^n$
- $A^{BC} = (A^B)^C$

Find the values for $c$ for which this converges.
(find the sum)

$$c - c^3 + c^5 - c^7 + c^9 - c^{11} + \ldots$$

$$\frac{-c^3}{c} = -c^2 \qquad \frac{c^5}{-c^3} = -c^2 \qquad \frac{-c^7}{c^5} = -c^2$$

Ratio $= -c^2$.

$\Rightarrow$ Geometric Series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$-1 < r < 1$$

The sum is

$a = c$, $r = -c^2$

$$\boxed{\frac{c}{1+c^2}}$$

$$-1 < -c^2 < 1$$

$$1 > c^2 > -1$$

always true

solve

$1 > c^2$ ✓

$$0 > c^2 - 1 = (c-1)(c+1)$$

$$\Rightarrow \boxed{-1 < c < 1}$$