MA163 Wk 6 Wed

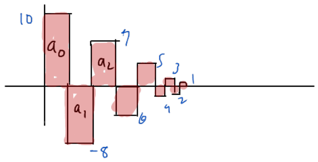
- Alternating Series Test
- Ratio test
- Root test

Alternating Series Test.

If Alternating: $\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \dots$

and $a_{n+1} < a_n$ (deeper in sequence $\Rightarrow$ less than previous) (decreasing terms)

then the series converges



$10 - 8 + 7 - 6 + 5 - 4 + 3 - 2 + 1 < 10$

10 2 9 3 8 4 7 5 6

Also: If $S = \sum_{n=0}^{\infty} (-1)^n a_n \Rightarrow S \leq a_0$

★ $|S - S_N| < a_{N+1}$

difference between what the series ACTUALLY converges to and it's N-th partial sum is less than the first omitted term

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

alternating harmonic series

① It's alternating $= \sum (-1)^{n+1} \cdot \left(\frac{1}{n}\right)$ (a_n)

② $\frac{1}{n+1} < \frac{1}{n}$ decreasing

$\Rightarrow$ A.S.T $\Rightarrow$ series converges

CONCEPTUAL INSIGHT

The convergence of an infinite series $\sum a_n$ depends on two factors: (1) how quickly a_n tends to zero, and (2) how much cancellation takes place among the terms. Consider:

Harmonic series (diverges) :	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$	— $\frac{1}{n}$ doesn't $\to 0$ fast enough
p-Series with $p = 2$ (converges) :	$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$	— $\frac{1}{n^2} \to 0$ FAST
Alternating harmonic series (converges):	$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$	$\to$ cancellation matters

The harmonic series diverges because reciprocals $1/n$ do not tend to zero quickly enough. By contrast, the reciprocal squares $1/n^2$ tend to zero quickly enough for the p-series with $p = 2$ to converge. The alternating harmonic series converges, but only due to the cancellation among the terms.

Since $|S - S_N| < a_{N+1} = \frac{1}{N+1}$

w/ $\sum (-1)^n \frac{1}{n}$

How far out do we have to go to be within $\frac{1}{1000}$ of the correct true #?

error b/w true # and N-th approx.

our error is $< \frac{1}{N+1}$ whenever we go out to N^{th} partial sum

set $\frac{1}{1000} = \frac{1}{N+1}$ solve for $N \Rightarrow N+1 = 1000$

$N = 999$

Ratio Test — {Applies to both positive & negative series} — Root Test

$$\sum_{n=0}^{\infty} a_n$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} <1 \Rightarrow \text{converge} \\ >1 \Rightarrow \text{diverge} \\ =1 \Rightarrow ??? \text{ inconclusive} \end{cases}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \begin{cases} <1 \Rightarrow \text{converge} \\ >1 \Rightarrow \text{diverge} \\ =1 \Rightarrow ??? \text{ inconclusive} \end{cases}$$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n}$

use Ratio test:

$$\lim_{n\to\infty} \frac{\frac{e^{n+1}}{n+1}}{\frac{e^n}{n}} = \lim_{n\to\infty} \frac{e^{n+1}}{n+1}\cdot\frac{n}{e^n}$$

$e^{n+1} = e^n\cdot e$

$$= \lim_{n\to\infty} \frac{e\cdot n}{n+1} = e > 1$$

$\Rightarrow$ diverges

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{(n+100)^n}{(n^2+1)^n} = 1 + \frac{101}{2} + \frac{(102)^2}{5^2} + \frac{(103)^3}{10^3} + \dots$

≈ 50

see powers $\Rightarrow$ Root Test

$$\lim_{n\to\infty} \sqrt[n]{\frac{(n+100)^n}{(n^2+1)^n}} = \frac{\sqrt[n]{(n+100)^n}}{\sqrt[n]{(n^2+1)^n}} = \lim_{n\to\infty} \frac{n+100}{n^2+1}$$

$= 0$

$\Rightarrow$ Converg

$5! = 5\cdot4\cdot3\cdot2\cdot1$

$$\frac{5!}{6!} = \frac{5\cdot4\cdot3\cdot4\cdot2\cdot1}{6\cdot5\cdot4\cdot3\cdot2\cdot1} = \frac{1}{6}$$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges

Apply Ratio Test:

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim \frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1 \Rightarrow \text{converge}$$