last one - alternating series test

1. Use the alternating series test to determine whether or not the following series are convergent.

(a)

$$\sum_{k=1}^{\infty} \frac{k \cos k\pi}{k^2+1} = \sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2+1}$$

Note the following:

$$(1) \lim_{k\to+\infty} \frac{k}{k^2+1} = \lim_{k\to+\infty} \frac{1}{2k} = 0 \Longrightarrow \left\{\frac{k}{k^2+1}\right\}_1^{+\infty} \text{ converges to } 0$$

$$(2) \frac{d}{dx}\left[\frac{x}{x^2+1}\right] = \frac{1-x^2}{(x^2+1)^2} \leq 0 \text{ for } x \geq 1 \Longrightarrow \left\{\frac{k}{k^2+1}\right\} \text{ is decreasing.}$$

Since the sequence of underlying terms, $\left\{\frac{k}{k^2+1}\right\}_1^{+\infty}$, is a decreasing sequence that converges to zero, the series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2+1}$ converges by the Althernating Series Test.

(b)

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k!}{(2k)!}$$

Consider the series without the alternating signs - i.e. does $\sum \frac{k!}{(2k)!}$ converge? Apply Ratio Test . . . .

$$\frac{a_{k+1}}{a_k} = \frac{\frac{(k+1)!}{(2(k+1))!}}{\frac{k!}{(2k)!}} = \frac{(k+1)!}{(2k+2)!} \cdot \frac{(2k)!}{k!} = \frac{(k+1)(k!)}{(2k+2)(2k+1)(2k!)} \cdot \frac{(2k)!}{k!}$$

$$= \frac{k+1}{(2k+2)(2k+1)} = \frac{1}{2(2k+1)}$$

$$\lim_{k\to+\infty} \frac{a_{k+1}}{a_k} \lim_{k\to+\infty} \frac{1}{2(2k+1)} = 0 < 1$$

Therefore

$$\sum_{k=1}^{\infty} \frac{k!}{(2k)!} \text{ converges by Ratio Test } \Longrightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1} k!}{(2k)!} \text{ converges}$$

(c)

$$\sum_{k=1}^{\infty} (-1)^{k+1} \sqrt[k]{k}$$

$$\text{Let } y = \lim_{k \to +\infty} \sqrt[k]{k} = \lim_{k \to +\infty} k^{1/k}.$$

$$\text{Then } \ln y = \ln \left[\lim_{k \to +\infty} k^{1/k} \right]$$

$$= \lim_{k \to +\infty} \ln \left[k^{1/k} \right] = \lim_{k \to +\infty} \frac{\ln k}{k} = \lim_{k \to +\infty} \frac{1/k}{1} = 0.$$

$$\text{So } y = \lim_{k \to +\infty} \sqrt[k]{k} = e^0 = 1 \neq 0$$

Therefore the series $\sum_{k=1}^{\infty} (-1)^{k+1} \sqrt[k]{k}$ diverges by the Divergence Test.

2. Give an example of . . .

a. an absolutely convergent alternating series.

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2}$$

b. a conditionally convergent alternating series.

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n}$$

c. a divergent alternating series.

$$\sum_{n=0}^{+\infty} (-1)^n$$

d. a divergent alternating series with $\lim_{n \to +\infty} a_n = 0$.

$$2, -1, 1, -\frac{1}{2}, \frac{2}{3}, -\frac{1}{3}, \frac{2}{4}, -\frac{1}{4}, \ldots$$