Math 163 - Calculus - Exam 2 - Guide **Name:** ____________________
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1. In each of the following, determine convergence/divergence. Indicate which test(s) you are using.

(1.1) Indicate absolute convergence, conditional convergence or divergence

$$\sum_{k=1}^{\infty} \frac{(-1)^n}{2n}$$

Alternating ✓

$\{\frac{1}{2n}\}$ decreasing (derivative $\frac{-1}{2n^2}$) By AST ⇒ series converges

$\sum \left|\frac{(-1)^n}{2n}\right| = \sum \{\frac{1}{n}\} \to \infty$

Harmonic Series

$\boxed{\text{Conditional Convergence}}$

(1.2) $$\sum_{k=2}^{\infty} \frac{2k\sqrt[3]{k}}{3k^2+5k+1}$$

L.C.T. w/ $\frac{2k\sqrt[3]{k}}{3k^2} = \frac{2}{3k^{2/3}}$,

$\lim \frac{2k^{4/3}}{3k^2+5k+1} \cdot \frac{3k^{2/3}}{2} = \lim \frac{6k^2}{6k^2+5k+1} = 1$ ⇒ basically same sequences

since $\sum \frac{2}{3k^{2/3}}$ diverges (p-test)

$\boxed{\text{this series diverges}}$

(1.3) $$\sum_{k=1}^{\infty} \cos\left(\frac{1}{k^2}\right)$$

L.C.T. $\lim_{k\to\infty} \frac{\cos\left(\frac{1}{k^2}\right)}{\frac{1}{k^2}} = \lim_{u\to 0} \frac{\cos(u)}{u} = \frac{1}{0} \to \infty$ ⇒ ?

$\boxed{\text{Div. Test: } \lim_{k\to\infty} \cos\left(\frac{1}{k^2}\right) = 1 \ \underline{\text{diverge}}}$

(1.4) $\sum_{k=1}^{\infty} \left[\frac{8}{5} - \frac{\sqrt[k]{5}}{2}\right]^k$

Root

$\lim \sqrt[k]{\left(\frac{8}{5} - \frac{\sqrt[k]{5}}{2}\right)^k} = \lim \frac{8}{5} - \frac{\sqrt[k]{5}}{2} = \frac{8}{5} - \frac{5^{1/\infty}}{2} = \frac{8}{5} - \frac{1}{2} = \frac{16-5}{10} = \frac{11}{10} > 1$ diverge by Root Test

(1.5) $\sum_{k=2}^{\infty} \frac{7k}{k^3+17}$

$\frac{7k}{k^3+17} < \frac{7k}{k^3} = \frac{7}{k^2}$

$\sum \frac{7}{k^2}$ converges $(p > 1)$

$\Rightarrow$ Comparison Test $\therefore$ converge

(1.6) $\sum_{k=0}^{\infty} \frac{5^{3k}}{(2k)!}$

ratio: $\frac{5^{3(k+1)}}{(2(k+1))!} \cdot \frac{(2k)!}{5^{3k}}$

$= \frac{5^{3k} \cdot 5^3}{(2k+2)(2k+1)(2k)!} \cdot \frac{2k!}{5^{3k}} = \frac{5^3}{(2k+2)(2k+1)} \longrightarrow 0 < 1$ $\boxed{\text{converge by ratio test}}$

(1.7) $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$

$\lim_{k\to\infty} \frac{\sin(\frac{1}{k})}{\frac{1}{k}} = \lim_{n\to 0} \frac{\sin(n)}{n} = 1 \Rightarrow$ terms grow at same rate

L.C.T. $\Rightarrow$ diverge since we know $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

Assume $\sum a_n$ converges. Thus, $\sum a_n = \lim_{n\to\infty} S_n = L$ (and $\lim_{n\to\infty} S_{n-1} = L$ too)

Recall, $a_n = S_n - S_{n-1}$

$\lim a_n = \lim S_n - S_{n-1} = 0$ ✓

2. Prove the following statement: If $\sum a_n$ converges, then $\lim_{n\to+\infty} a_n = 0$

3. Find the value of the convergent series below:

(3.1) $\sum_{k=1}^{+\infty} \frac{2^{k+1}}{3^{k-1}} = \frac{2 \cdot 2^k}{3^{-1} 3^k} = 6 \cdot \left(\frac{2}{3}\right)^k = \frac{6}{1/3} = 18$

(3.2) $\sum_{k=2}^{+\infty} \left[64^{1/k} - 64^{1/(k+2)}\right]$

$S_2 = 64^{1/2} - 64^{1/4}$

$S_3 = 64^{1/2} - 64^{1/4} + 64^{1/3} - 64^{1/5}$

$S_4 = 64^{1/2} - 64^{1/4} + 64^{1/3} - 64^{1/5} + 64^{1/4} - 64^{1/6} = 64^{1/2} + 64^{1/3} - 64^{1/5} - 64^{1/6}$

$S_5 = 64^{1/2} + 64^{1/3} - 64^{1/5} - 64^{1/6} + 64^{1/5} - 64^{1/7} = 64^{1/2} + 64^{1/3} - 64^{1/6} - 64^{1/7}$

$S_k = \underbrace{\underset{8}{\underset{\shortparallel}{64^{1/2}}} + \underset{4}{\underset{\shortparallel}{64^{1/3}}}}_{12} - 64^{\frac{1}{k+1}} - 64^{\frac{1}{k+2}}$,

$\text{Series} = \lim_{k\to\infty} S_k = 12 - \lim_{k\to\infty} 16^{\frac{1}{k+1}} - \lim_{k\to\infty} 16^{\frac{1}{k+2}} = 12 - 16^0 - 16^0 = 10$

4. Give three examples (each) of . . .

(4.1) a divergent alternating series

$-1+1-1+1-1+1\ldots = \sum (-1)^n$

$\sum 3(-1)^n + 5$

$\sum n(-1)^n$

(4.2) a conditionally convergent alternating series.

$\sum k\frac{(-1)^n}{n}$

$k=1,2,3$

(4.3) an absolutely convergent alternating series

$\sum k \cdot \frac{(-1)^n}{n^2}$

$k=1,2,3$

(4.4) a decreasing *sequence* that converges to $\ln 7$.

$\left\{\ln 7 + \frac{k}{n}\right\}$, $k=1,2,3$

(4.5) a strictly increasing *sequence* that converges to e.

$\left\{e - \frac{k}{n}\right\}$ $k=1,2,3$

or

$\left\{\frac{n}{n+1}\cdot e\right\}_{n=1}^{\infty}$