Warm-up

Determine conv/div

(1 $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$

Fact: $\frac{1}{n^2+n+1} \leq \frac{1}{n^2}$ ($\forall$ for all $n \geq 1$)

Because
$n^2+n+1 \geq n^2$
$\Updownarrow$ cross divide
$\frac{1}{n^2} \geq \frac{1}{n^2+n+1}$

Since $\sum \frac{1}{n^2}$ converges ($p>1$)
$\Rightarrow \sum \frac{1}{n^2+n+1}$ converges

Hint: Comparison Test

1. guess conv or div

Find a convergent series whose terms are upper bound
$\frac{1}{n^2}, \frac{1}{n^3}, \ldots$

Find a divergent series that is "beneath" the given one
$\frac{1}{n}, \frac{1}{\sqrt{n}}, \frac{1}{n^{1/3}}$

(2 $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

Since $\frac{1}{2^n+1} \leq \frac{1}{2^n}$ & $\sum \frac{1}{2^n}$ converges (due to it being a geometric series w/ $r=\frac{1}{2}$)

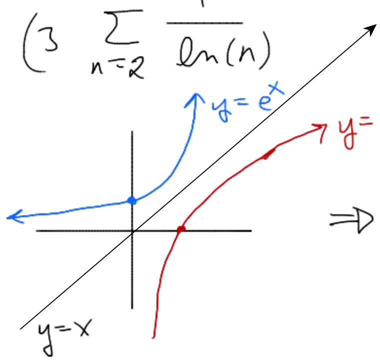
we conclude that $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

Given $\sum a_n = \sum_{n=1}^{\infty} a_n$

If $a_n \leq b_n$
AND $\sum b_n$ converges
Then $\sum a_n$ converges

If $b_n \leq a_n$
AND $\sum b_n$ diverges
then $\sum a_n$ diverges

(3 $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$



$\Rightarrow$ Fact: $\boxed{\ln(x) < x}$

i.e., $\ln\left(\frac{1}{x}\right) < \frac{1}{x}$

cross div

$\frac{1}{x} < \frac{1}{\ln x}$

So

$\frac{1}{\ln(n)} > \frac{1}{n}$ $\forall n$

& $\sum \frac{1}{n}$ diverges So $\sum \frac{1}{\ln(n)}$ diverges

Today: Limit Comparison Test

For $\sum_{n=1}^{\infty} \frac{1}{n+1}$ comparing it to $\frac{1}{n}$ leads to:

$$\frac{1}{n+1} < \frac{1}{n}$$
$$\downarrow$$
$\sum \frac{1}{n}$ diverges

$\Rightarrow$ This tells us nothing about conv/div of $\sum \frac{1}{n+1}$

In situations like this: use the Limit Comparison Test:

For $\sum a_n$, $\sum b_n$ that are given

if $\frac{a_n}{b_n} \longrightarrow L \neq 0$ then the series have the same convergence / divergence

($\sum a_n$ converges $\longleftrightarrow$ $\sum b_n$ converges
diverges $\longleftrightarrow$ diverges)

if $\frac{a_n}{b_n} \longrightarrow 0$ $\frac{a_n}{b_n}$ are bounded, i.e., $\frac{a_n}{b_n} \leq M \;\forall n$

so $a_n \leq M \cdot b_n$

apply comparison test:
if $\sum M b_n$ converges (equivalently $\sum b_n$ converges)
then $\sum a_n$ converges

if $\frac{a_n}{b_n} \longrightarrow \infty$ $\frac{a_n}{b_n} > M$ (fixed constant) for $n \geq N$ $\Rightarrow$ $a_n > b_n M$

apply comparison test:
if $\sum M b_n$ (equiv. $\sum b_n$) diverges
then $\sum a_n$ diverges.

Ex $\sum \frac{5^n}{3^n+2}$

might try:

$$\frac{5^n}{3^n+2} \le \frac{5^n}{3^n} = \left(\frac{5}{3}\right)^n$$

$\sum \left(\frac{5}{3}\right)^n$ $\rightarrow$ diverges

(inconclusive comp. test)

$$a_n = \frac{5^n}{3^n+2} \qquad b_n = \frac{5^n}{3^n}$$

$$\frac{a_n}{b_n} = \frac{\frac{5^n}{3^n+2}}{\frac{5^n}{3^n}} = \frac{3^n}{5^n} \cdot \frac{5^n}{3^n+2} = \frac{3^n}{3^n+2} \xrightarrow{n \to \infty} \frac{3^n \cdot \ln 3}{3^n \ln 3} = 1$$

$\Rightarrow$ same conv/div: $\sum \frac{5^n}{3^n}$ diverges (geometric $r > 1$)

$\Rightarrow$ given series diverges by L.C.T.

Thursday - Week 7

Warm-up: Determine the convergence/divergence of:

① $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$

Hint: Comparison Test

if $a_n \le b_n$
AND $\sum b_n$ converges
then $\sum a_n$ converge

if $b_n \le a_n$
AND $\sum b_n$ diverges
then $\sum a_n$ diverge

Fact: $\forall n \geq 1$

$\frac{1}{n^2+n+1} \le \frac{1}{n^2}$

B/C $n^2+n+1 > n^2$ $\forall n \geq 1$ (for all)
$\Rightarrow \frac{1}{n^2} > \frac{1}{n^2+n+1}$

& $\sum \frac{1}{n^2}$ converges (p-test w/ $p=2$)

$\Rightarrow$ Comp. Test shows $\sum \frac{1}{n^2+n+1}$ converges

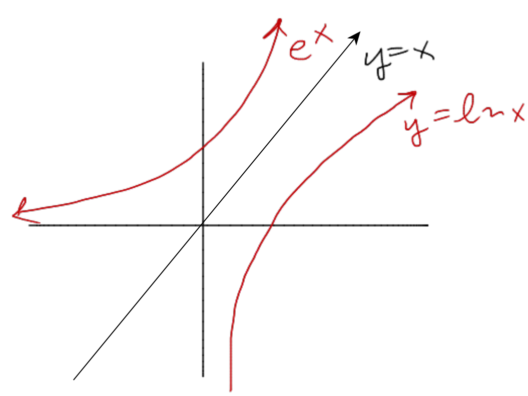
② $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

Fact: $\frac{1}{2^n+1} \le \frac{1}{2^n}$

$\sum \left(\frac{1}{2^n}\right) = \sum \left(\frac{1}{2}\right)^n$ converges
geometric
$a=1$
$r=\frac{1}{2}$

Comparison Test $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

③ $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$



$\frac{1}{\ln(n)} > \frac{1}{n}$

B/C $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\Rightarrow \sum \frac{1}{\ln(n)}$ diverges (Comparison Test)

$x > \ln x$

$\frac{1}{\ln x} > \frac{1}{x}$

Try to use Comp Test for

$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

① p-test $p>1 \Leftrightarrow$ Convergence

② $\int_1^\infty \frac{1}{x} = \ln x \Big|_1^\infty = \infty$

Idea #1 $\frac{1}{n+1} < \frac{1}{n}$ but $\sum \frac{1}{n}$ diverges inconclusive

Idea #2 $\frac{1}{n+1} < \frac{n}{n+1}$ $(\forall\, n \geq 1$ but $\sum \frac{n}{n+1}$ diverges (Divergence: $\lim_{n\to\infty} \frac{n}{n+1} = 1 \Rightarrow$ diverges

Limit Comparison Test

If this happens: use this!

Idea: Given $\sum a_n$. Compare to $\sum b_n$.

If $\frac{a_n}{b_n} \xrightarrow{n\to\infty} L \neq 0$ (i.e., after some index: $\frac{a_n}{b_n} \approx L$ (if n is sufficiently large

$a_n \approx L \cdot b_n$

same convergence / divergence)

If $\frac{a_n}{b_n} \longrightarrow 0$ (All but finitely many terms of $\frac{a_n}{b_n}$ are close to 0. thus $\frac{a_n}{b_n} < M$ Fixed Constant

$a_n < M \cdot b_n$

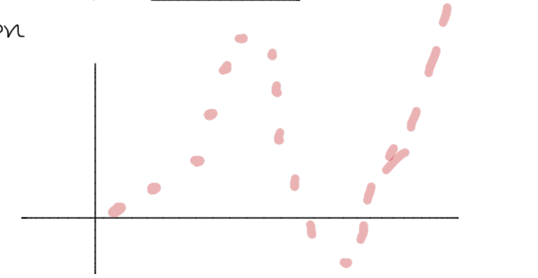
Comparison Test $\Rightarrow$ If $\sum M b_n$ or $\sum b_n$ converges then $\sum a_n$ converges

If $\frac{a_n}{b_n} \longrightarrow \infty$ $\quad \frac{a_n}{b_n} > M$ fixed constant after some index

$a_n > M \cdot b_n$

apply comparison test

If b_n diverges then a_n diverges



Example:

$$\sum_{n=1}^{\infty} \frac{5^n}{3^n + 2}$$

L.C.T.

$$\frac{\frac{5^n}{3^n+2}}{\frac{5^n}{3^n}} = \frac{3^n}{5^n} \cdot \frac{5^n}{3^n+2} = \frac{3^n}{3^n+2}$$

$$\Big\downarrow n \to \infty$$

$$1$$

Try:

$$\frac{5^n}{3^n+2} < \frac{5^n}{3^n}$$

$\sum \frac{5^n}{3^n}$ divergences b/c $\underline{r > 1}$

$= \sum \left(\frac{5}{3}\right)^n$

geometric $= \sum a r^n$

$a = 1$

$r = 5/3$

B/c $\lim \frac{a_n}{b_n} = 1$

same conv/div.

$\sum \frac{5^n}{3^n}$ diverges

So, $\sum \frac{5^n}{3^n+2}$ diverges

$$\text{Grade} = .25(\underset{\uparrow}{\overset{\text{avg.}}{\text{WeBWork}}}) + .50(\underset{\uparrow}{\overset{\text{avg}}{\text{Exams}}}) + .25(\text{Final})$$

$$= .25(70) + .5(72) + .25(80)$$

$$\sum_{n=1}^{20} \frac{1}{n^p} = \frac{1}{3^{1/3}} + \frac{1}{4^{1/3}} + \frac{1}{5^{1/3}} + \dots + \frac{1}{10^{1/3}} + \dots + \frac{1}{20^{1/3}}$$

Comparison: $a_n \le b_n$ $\quad \sum b_n$ conv $\Rightarrow \sum a_n$ conv

$b_n \le a_n$ $\quad \sum b_n$ div $\Rightarrow \sum a_n$ div

Limit Comparison Test

$\frac{a_n}{b_n} \longrightarrow 0$ if $\sum b_n$ conv, $\sum a_n$ conv.

$\frac{a_n}{b_n} \longrightarrow L \neq 0$ (same conv/div b/w a_n & b_n

$\frac{a_n}{b_n} \longrightarrow \infty$ if $\sum b_n$ div then $\sum a_n$ div.