Wed. Week 7

warm-up:

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$$

Hint: Integral Test : The conv/div of the series is equal to the conv/div of an integral

Let

$f(x) = \frac{x}{(x^2+1)^2}$, $f(n) = \frac{n}{(n^2+1)^2}$, if $f(x)$ is cts, decreasing, pos. $x \geq 1$

then the conv/div of the series is tied to this

$$\int_1^{\infty} f(x)\,dx$$

Observe:

① f is cts b/c rational fcns are cts

② $f(x)$ is positive

$$\frac{x}{(x^2+1)^2} > 0 \quad \text{when } x \geq 1$$

③ decreasing: show $f' < 0$

$$f'(x) = \frac{(x^2+1)^2 \cdot 1 - x(2(x^2+1) \cdot 2x)}{(x^2+1)^4} < 0 \quad \text{for } x \geq 1$$

$$= \frac{(x^2+1)^2 - 4x^2(x^2+1)}{(x^2+1)^4}$$

$$= \frac{(x^2+1)(x^2+1-4x^2)}{\sim}$$

$$= \frac{(x^2+1)(1-3x^2)}{\sim} < 0 \quad \text{for } x \geq 1$$

$$\frac{(+)(-)}{(+)} = (-) < 0$$

$$\frac{1}{2}\int_{x=1}^{x=\infty} \frac{2x}{(x^2+1)^2}\,dx$$

$u = x^2+1$
$du = 2x\,dx$

$$\frac{1}{2}\int_2^{\infty} \frac{du}{u^2} = \frac{1}{2}\left(\frac{-1}{u}\right)\Bigg|_2^{\infty}$$

$$= \frac{1}{2}\left[\underbrace{\frac{-1}{\infty}}_{=0} - \frac{-1}{2}\right] = \frac{1}{4}$$

integral converges

$\Rightarrow$ $\boxed{\text{series converges}}$

What we know about series ———
today

1. geometric series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ $\quad |r|<1$

2. divergence test: $\lim_{n\to\infty} a_n \neq 0$ then series diverges

3. integral test:

Today: p-test, comparison test:

p-test: Concerns $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad$ (p = power of n as it sits downstairs)

$p=3 \Rightarrow \frac{1}{n^3} = n^{-3}$

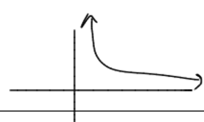
You know the conv / div for:

$p=1$ $\quad \sum_{n=1}^{\infty} \frac{1}{n} \xrightarrow{\text{Harmonic}}$ diverges

$p=2$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^2} \longrightarrow$ converges to $\frac{\pi^2}{6}$

Theorem: (if and only if)

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges exactly when $p>1$



proof:

let $f(x) = \frac{1}{x^p}$, cts, pos., decreasing $(x \geq 1)$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \left.\frac{x^{1-p}}{1-p}\right|_1^{\infty} = \begin{cases} \frac{x^{1-(-2)}}{1-(-2)} = \frac{x^3}{3} \to \infty \\ \left.\frac{1}{1-p}\cdot\frac{1}{x^{p-1}}\right|_1^{\infty} \end{cases}$$

$p=1$: $\int_1^{\infty} \frac{1}{x}\,dx = \ln|x|\Big|_1^{\infty} = \infty$

$p \neq 1$

$$= \frac{1}{1-p}\Big[\underbrace{\tfrac{1}{\infty}}_{=0} - \underbrace{\tfrac{1}{1}}_{=1}\Big] = \boxed{\frac{1}{p-1}}$$ converges

say $p=-2$

if $p<1$ then $0<1-p$

if $p>1$ then $0>1-p$

say $p>2$

Determine conv/div

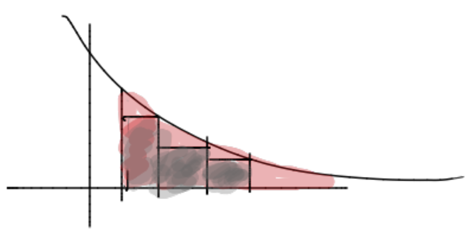
<u>Ex</u> $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ <u>p-test</u> $p = \frac{1}{2} < 1 \Rightarrow$ diverges

<u>Ex</u> $\sum_{n=1}^{\infty} \frac{1}{n^7}$ p-test: $p = 7 > 1 \Rightarrow$ converges

$$\longleftarrow \frac{1}{n^{-1}} \text{———} \frac{1}{\sqrt{n}} \text{———} \frac{1}{n} \longrightarrow \frac{1}{n^{10}}$$

$\frac{1}{n^{-1}} = n$

converge



Consider:

$$\sum_{n=1}^{\infty} \frac{n+1}{n^6}$$

Not: integral test

p-series $\sum \frac{1}{n^6}$

Compare to a known series

$$\frac{n+1}{n^6} \geq \boxed{\frac{n}{n^6}}$$

$$= \frac{1}{n^5} \qquad \boxed{\sum \frac{1}{n^5}} \downarrow \text{converges}$$

If the lower series diverges then the upper series diverges too

or

$$\frac{n^2}{n^6} \geq \frac{n+1}{n^6}$$

true for all $n \geq 2$

$$\frac{1}{n^4} \text{ or } \boxed{\sum \frac{1}{n^4}}$$

$\Downarrow$

converges

If the upper series converges then the lower series converge

$\Rightarrow$ our given series converges