Exam 2 Guide - Solutions

1. Determine convergence/divergence. Indicate which test(s) you are using. Answer for at least 5 of the 7 series below.

(a) Apply Alternating Series Test

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k}$$

1. $\lim\limits_{k\to+\infty} \frac{1}{2k} = 0,$ and

2. $\frac{d}{dx}\left[\frac{1}{2x}\right] = -\frac{1}{2}x^{-2} < 0$ for all $x > 0$

$\implies f(x) = \frac{1}{2x}$ is a strictly decreasing function for $x > 0$

$$\implies \left\{\frac{1}{2k}\right\}_{k=1}^{+\infty} \text{ is decreasing}$$

Therefore the series converges by the Alternating Series Test.

(b) Apply Limit Comparison Test, use $\sum \frac{1}{k^{2/3}}$

$$\sum_{k=2}^{\infty} \frac{2k\sqrt[3]{k}}{3k^2+5k+1}$$

Consider $\sum \frac{1}{k^{2/3}}$, a divergent p-series, $p = 2/3$.

$$\lim_{k\to+\infty} \frac{\frac{2k\sqrt[3]{k}}{3k^2+5k+1}}{\frac{1}{k^{2/3}}} = \lim_{k\to+\infty} \frac{2k^2}{3k^2+5k+13} \stackrel{\text{L'H}}{=} \lim_{k\to+\infty} \frac{4k}{6k+5} \stackrel{\text{L'H}}{=} \lim_{k\to+\infty} \frac{4}{6} = \frac{2}{3}$$

Since this limit is finite and nonzero, the series $\sum \frac{2k\sqrt[3]{k}}{3k^2+5k+1}$ diverges by the Limit Comparizon Test.

(c) Apply Divergence Test

$$\sum_{k=1}^{\infty} \cos\left(\frac{1}{k^2}\right)$$

$$\lim_{k\to+\infty} \cos\left(\frac{1}{k^2}\right) = 1 \neq 0$$

Therefore the series diverges by the Divergence Test.

(d) Apply Root Test

$$\sum_{k=1}^{\infty}\left[\frac{8}{5}-\frac{\sqrt[k]{5}}{2}\right]^k$$

$$\sqrt[k]{a_k}=\sqrt[k]{\left[\frac{8}{5}-\frac{\sqrt[k]{5}}{2}\right]^k}=\left[\frac{8}{5}-\frac{\sqrt[k]{5}}{2}\right]$$

$$\lim_{k\to+\infty}\sqrt[k]{a_k}=\lim_{k\to+\infty}\left[\frac{8}{5}-\frac{\sqrt[k]{5}}{2}\right]=\frac{8}{5}-\frac{1}{2}=\frac{11}{10}>1$$

Therefore the series diverges by the Root Test.

(e) Apply Limit Comparison Test, use $\sum \frac{1}{k^2}$

$$\sum_{k=2}^{\infty}\frac{7k}{k^3+17}$$

Consider $\sum \frac{1}{k^2}$, a convergent p-series, $p=2$.

$$\lim_{k\to+\infty}\frac{\frac{7k}{k^3+17}}{\frac{1}{k^2}}=\lim_{k\to+\infty}\frac{7k^3}{k^3+17}\overset{\text{L'H}}{=}\lim_{k\to+\infty}\frac{21k^2}{3k^2}=\lim_{k\to+\infty}7=7$$

Since this limit is finite and nonzero, the series $\sum \frac{7k}{k^3+17}$ converges by the Limit Comparizon Test.

(f) Apply Ratio Test

$$\sum_{k=0}^{\infty}\frac{5^{3k}}{(2k)!}$$

$$\frac{a_{k+1}}{a_k}=\frac{\frac{5^{3(k+1)}}{(2(k+1))!}}{\frac{5^{3k}}{(2k)!}}=\frac{5^{3k+3}}{(2k+2)!}\cdot\frac{(2k)!}{5^{3k}}=\frac{5^{3k}\cdot 5^3}{(2k+2)(2k+1)(2k)!}\cdot\frac{(2k)!}{5^{3k}}$$

$$=\frac{125}{(2k+2)(2k+1)}$$

$$\lim_{k\to+\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to+\infty}\frac{125}{(2k+2)(2k+1)}=0<1$$

Therefore the series converges by the Ratio Test.

(g) Apply Limit Comparison Test, use $\sum \frac{1}{k}$

$$\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$$

Consider $\sum \frac{1}{k}$, a dinvergent p-series, $p = 1$.

$$\lim_{k\to+\infty} \frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} \overset{\text{L'H}}{=} \lim_{k\to+\infty} \frac{-k^{-2}\cos\left(\frac{1}{k}\right)}{-k^{-2}} = \lim_{k\to+\infty} \cos\left(\frac{1}{k}\right) = 1$$

Since this limit is finite and nonzero, the series $\sum \sin\left(\frac{1}{k}\right)$ diverges by the Limit Comparizon Test.

2. Prove the following statement:

$$\text{If } \sum a_n \text{ converges, then } \lim_{n\to+\infty} a_n = 0$$

Assume $\sum a_n$ converges. Then by definition, there is a finite value L such that

$$\sum a_n = \lim_{n\to+\infty} s_n = L.$$

Note that $a_n = s_n - s_{n-1}$. Then

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} [s_n - s_{n-1}] = L - L = 0.$$

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3. Find the value of the convergent series below:
(a) Geometric series, $r = 2/3$ and initial value 4

$$\sum_{k=1}^{+\infty} \frac{2^{k+1}}{3^{k-1}} = 4 + \frac{8}{3} + \frac{16}{9} + \cdots = \frac{4}{1-\frac{2}{3}} = 12$$

(b)

$$\sum_{k=2}^{+\infty}\left[64^{1/k}-64^{1/(k+2)}\right]$$

$$a_2=64^{\frac{1}{2}}-64^{\frac{1}{4}}, a_3=64^{\frac{1}{3}}-64^{\frac{1}{5}}, a_4=64^{\frac{1}{4}}-64^{\frac{1}{6}}, \ldots$$

$$s_2=a_2=64^{\frac{1}{2}}-64^{\frac{1}{4}}$$

$$s_3=s_2+a_3=64^{\frac{1}{2}}+64^{\frac{1}{3}}-64^{\frac{1}{4}}-64^{\frac{1}{5}}$$

$$s_4=s_3+a_4=64^{\frac{1}{2}}+64^{\frac{1}{3}}-64^{\frac{1}{5}}-64^{\frac{1}{6}}$$

$$\Longrightarrow s_k=64^{\frac{1}{2}}+64^{\frac{1}{3}}-64^{\frac{1}{k+1}}-64^{\frac{1}{k+2}}$$

Then

$$\sum_{k=2}^{+\infty}\left[64^{1/k}-64^{1/(k+2)}\right]=\lim_{k\to+\infty}s_k$$

$$=\lim_{k\to+\infty}\left[64^{\frac{1}{2}}+64^{\frac{1}{3}}-64^{\frac{1}{k+1}}-64^{\frac{1}{k+2}}\right]=8+4-1-1=10$$

4. Give three examples (each) of . . .

(a) a divergent alternating series

$$\sum_{n=0}^{+\infty}(-1)^n\,,\ \sum_{n=0}^{+\infty}(-1)^n n\,,\ \sum_{n=0}^{+\infty}(-1)^n n^2$$

(b) a conditionally convergent alternating series.

$$\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}\,,\ \sum_{n=1}^{+\infty}\frac{(-1)^n}{\sqrt{n}}\,,\ \sum_{n=1}^{+\infty}\frac{(-1)^n}{\sqrt[3]{n}}$$

(c) an absolutely convergent alternating series

$$\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\,,\ \sum_{n=1}^{+\infty}\frac{(-1)^n}{n^3}\,,\ \sum_{n=1}^{+\infty}\frac{(-1)^n}{n^4}$$

(d) a decreasing *sequence* that converges to $\ln 7$.

$$\left\{\ln 7+\frac{1}{n}\right\}_{n=1}^{+\infty}, \left\{\ln 7+\frac{1}{n^2}\right\}_{n=1}^{+\infty}, \left\{\ln 7+\frac{1}{\sqrt{n}}\right\}_{n=1}^{+\infty}$$

(e) a strictly increasing *sequence* that converges to e.

$$\left\{e-\frac{1}{n}\right\}_{n=1}^{+\infty}, \left\{e-\frac{1}{n^2}\right\}_{n=1}^{+\infty}, \left\{e-\frac{1}{\sqrt{n}}\right\}_{n=1}^{+\infty}$$