MA163 wk 8 - Fri

10.7 (Taylor, Maclaurin Series)

warm-up: Remarkable Formula for π

start.

$$\boxed{\frac{1}{1+x^2}} = \frac{1}{1-(-x^2)} = \frac{1}{1-u} = 1+u+u^2+u^3+u^4+\dots$$

sub $-x^2$

$$\boxed{= 1-x^2+x^4-x^6+x^8-x^{10}+\dots} \qquad |x|<1$$

integrate both sides

$$\int \frac{1}{1+x^2}dx = \boxed{\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{10}-\frac{x^{11}}{11}}$$

sub $x=1$

tan(x) is just the slope corresponding to angle x

arctan(x) is just the angle corresponding to slope x

$$\frac{\pi}{4} = \tan^{-1}(1) = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\dots$$

$$\pi = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\dots\right) \quad \text{(Liebniz)}$$

Motivation For Taylor Series (and Maclaurin)

If $\boxed{f(x) = \sum_{n=0}^{\infty} a_n x^n}$ centered @ $x = 0$.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

$f(0) = a_0$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4$$

$f'(0) = a_1$

$$f''(x) = 2a_2 + 3 \cdot 2 a_3 x + 4 \cdot 3 a_4 x^2 + 5 \cdot 4 \cdot a_5 x^3 +$$

$\boxed{f''(0) = 2a_2}$

$$f'''(x) = 3 \cdot 2\, a_3 + 4 \cdot 3 \cdot 2\, a_4 x + 5 \cdot 4 \cdot 3\, a_5 x^2 + \ldots$$

$f'''(0) = 3 \cdot 2 \cdot a_3$

$a_0 = f(0)$

$a_1 = f'(0)$

$a_2 = \dfrac{f''(0)}{2}$

$a_3 = \dfrac{f'''(0)}{3 \cdot 2}$

$a_4 = \dfrac{f^{(iv)}(0)}{4!}$

$\boxed{a_n = \dfrac{f^{(n)}(0)}{n!}}$

so:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$ Maclaurin Series (centered @ $x = 0$)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$ (centered @ $x = c$)

Taylor Series

HW1 Part 1 #1

Write out the first four terms of the Maclaurin series of f if:

$$f(0)=7,\ f'(0)=7,\ f''(0)=18,\ f'''(0)=18$$

$$\boxed{\text{Maclaurin Series} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^n}$$

$$= \frac{f^{0}(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$7 = f(0) = f^0(0)$, $f'(0) = 7$, $f''(0) = 18$, $f'''(0) = 18$

$$= 7 + 7x + 9x^2 + 3x^3$$

Ex

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) x^n$$

$$\frac{1}{n!} = \frac{f^{(n)}(0)}{n!}$$

$$f^{(n)}(0) = \frac{d^n}{dx^n}(e^x) = e^x \Big|_{x=0} = 1$$

Maclaurin Series for cos(x)
$\Rightarrow$ center @ $x=0$

$f^4 = f(x) = \cos(x)$ $\qquad f(0) = \cos(0) = 1$

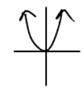
$f^5 = f'(x) = -\sin(x)$ $\qquad f'(0) = -\sin(0) = 0$

$f^6 = f''(x) = -\cos(x)$ $\qquad f''(0) = -\cos(0) = -1$

$f^7 = f'''(x) = \sin(x)$ $\qquad f'''(0) = \sin(0) = 0$

$f^{(IV)}(x) = \cos x$ $\qquad f^{IV}(0) = 1$

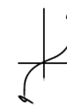
$f(-x) = f(x)$



even

$$\text{MacL} = f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \underbrace{\frac{1}{0!}}_{=1} x^0 + \underbrace{\frac{0}{1!}}_{=0} x^1 + \frac{(-1)}{2!} x^2 + \underbrace{\frac{0}{3!}}_{=0} x^3 + \frac{1}{4!} x^4$$



odd

$f(-x) = -f(x)$

Taylor series @ $x=0$ up to $n=4$

$$T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$$

For sin(x):

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$