10.7 Taylor Polynomials
(and MacLaurin)

Happy π-day!

Remarkable Formula For π.

start
(1) $$\frac{1}{1+x^2} = \underbrace{\frac{1}{1-(-x^2)}}_{\frac{1}{1-u} = 1+u+u^2+u^3+u^4+\cdots} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \qquad (|x|<1)$$

(2) Integrate Both Sides

$$\tan^{-1}x = \int \frac{1}{1+x^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

(3) sub $x=1$:

$$\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots$$

the angle needed to make a slope of 1

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots\right)$$

Motivation For Taylor & MacLaurin Polynomials

Suppose $f(x)$ is expressed as a power series (centered @ $x=0$)

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n x^n} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

then $f(0) = a_0$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots$$

so $f'(0) = a_1$

$$f''(x) = 2a_2 + 3\cdot 2 a_3 x + 4\cdot 3 a_4 x^2 + 5\cdot 4 \cdot a_5 x^3$$

so $f''(0) = 2a_2$

$$f'''(x) = 3\cdot 2 a_3 + 4\cdot 3\cdot 2 x + 5\cdot 4\cdot 3 a_5 x^2$$

so $f'''(0) = 3\cdot 2 a_3$

etc...

Recap: Solve For a_i $\Big\}$ $a_0 = f(0)$, $a_1 = f'(0)$ | $a_2 = \dfrac{f''(0)}{2}$, $a_3 = \dfrac{f'''(0)}{3\cdot 2}$ | $a_4 = \dfrac{f^{(IV)}(0)}{4!}$, $a_5 = \dfrac{f^{(V)}(0)}{5!}$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\cdot x^n}$$ = MacLaurin Polynomial

↑ centered @ 0

(If centered @ $x=c$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}\cdot (x-c)^n}$$ Taylor Polynomial

Ex (HW 1 Part 1, #1)

Write out the first <u>four</u> terms of the Maclaurin series of f(x) if

$$f(0)=7, \quad f'(0)=7, \quad f''(0)=18, \quad f'''(0)=18.$$

Maclaurin: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad \left(f^{(0)}(x) = f(x)\right)$

$$= \frac{f^{0}(0)}{0!}x^0 + \frac{f'(0)}{1}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 7 + 7x + 9x^2 + 3x^3$$

Calculate the Taylor Polynomials $T_2(x)$, $T_3(x)$, $T_4(x)$ centered @ $x=0$ for $f(x) = \cos(x)$.

$\Rightarrow$ Maclaurin

$$" \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^n = \overbrace{\underbrace{f(0) + f'(0)x + \frac{f''(0)x^2}{2!}}_{T_2} + \frac{f'''(0)x^3}{3!}}^{T_3} + \frac{f^{(IV)}(0)x^4}{4!}$$

$f(x) = \cos(x)$ | $f(0) = 1$

$f'(x) = -\sin(x)$ | $f'(0) = 0$

$f''(x) = -\cos(x)$ | $f''(0) = -1$

$f'''(x) = \sin(x)$ | $f'''(0) = 0$

$f^{(IV)}(x) = \cos(x)$ | $f^{(IV)}(0) = 1$

$$= 1 + 0x + \frac{(-1)x^2}{2!} + \frac{0}{3!}x^3 + \frac{1 \cdot x^4}{4!}$$

T_4

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

(in general

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

$\underline{Ex}$ Binomial Expansion

$$(1+x)^{-4/3}$$

$$\boxed{\binom{a}{n} = \frac{a!}{n!\,(a-n)!}}$$

Nice Formula For coef.

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$$

$$\binom{a}{0} = \frac{a!}{0!\,(a!)} = 1$$