10.6 / 10.7 Power Series: "manipulations"

Key Idea: $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ when this converges, manipulations are valid

substitutions
derivatives
integrals

$(x-1)^3$
$\downarrow d/dx$
$3(x-1)^2$

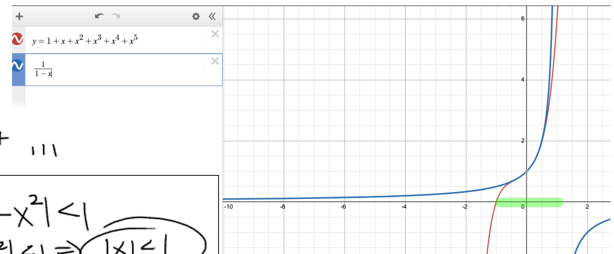
e.g. $f(3) = \underbrace{\sum_{n=0}^{\infty} a_n (3-c)^n}_{\text{some \#}}$ | $f(-x^2) = \underbrace{\sum_{n=0}^{\infty} a_n (-x^2-c)^n}_{\text{new series}}$ | $\frac{d f(x)}{dx} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right)$

$" = \sum_{n=0}^{\infty} a_n \cdot n (x-c)^{n-1}$

Substitutions:

"geometric"

Replace x w/ $-x^2$ in $\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots \text{ when } |x|<1}$



$= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + (-x^2)^5 + \dots$

$$\boxed{\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots \quad \text{when } |-x^2|<1 \text{ or } |x^2|<1 \Rightarrow |x|<1}$$

we'll see π = infinite series

$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots\right)$

EX write a power series representation for & give its interval of convergence

$f(x) = \frac{1}{2+x^2}$

manipulate to match some function w/ a known power series, eg $\frac{1}{1-x}$

① $\frac{1}{2+x^2} = \frac{1}{2(1+\frac{x^2}{2})} = \frac{1}{2(1-(-\frac{x^2}{2}))}$

② set $u = -\frac{x^2}{2}$ $= \frac{1}{2(1-u)} = \frac{1}{2} \cdot \frac{1}{1-u}$ // know power series

③ $= \frac{1}{2}\sum_{n=0}^{\infty} u^n = \underbrace{\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-x^2}{2}\right)^n}_{\text{new power series for } \frac{1}{2+x^2}} = \frac{1}{2}\sum(-1)^n\cdot\left(\frac{x^2}{2}\right)^n = \frac{1}{2}\sum(-1)^n\cdot\frac{x^{2n}}{2^n}$

$= \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2^{n+1}}}$

valid when $|u|<1$

$\left|\frac{-x^2}{2}\right|<1$

$\left|\frac{x^2}{2}\right|<1$

④ $-1 < \frac{x^2}{2} < 1$

$\underbrace{-2<}_{\text{always}} x^2 < 2$ square root $|x| < \sqrt{2}$ $\Rightarrow$ $\boxed{x \in (-\sqrt{2}, \sqrt{2})}$

⑤ Check Endpts

sub: $x=\sqrt{2}$ into OG. $\frac{1}{2}\sum\left(-\frac{(\sqrt{2})^2}{2}\right)^n = \frac{1}{2}\sum(-1)^n$ div $\Rightarrow$ exclude

similarly for $x = -\sqrt{2}$

Derivatives of (convergent) Power Series

$\frac{1}{1-x} = (1-x)^{-1}$

$-1(1-x)^{-2}(-1)$

Start w/

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\Big\downarrow \tfrac{d}{dx}$$

$$\boxed{\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n}$$

Ex: $\frac{1}{(1-(.1))^2} \approx 1 + 2(.1) + 3(.1)^2 = 1 + .2 + .03 = 1.23$

$\left(\frac{1}{10}\right)^2 = \frac{1}{100}$

Recall, in Calculus I, you saw only one function that equals its own derivative

eg.

$\sin x \xrightarrow{d/dx} \cos x$

$\frac{1}{x} \longrightarrow \frac{-1}{x^2}$

$\cos x \longrightarrow -\sin x$

$e^{2x} \longrightarrow 2e^{2x}$

$e^x \longrightarrow e^x$

Start $\sum \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$\downarrow d/dx$ $\quad \frac{\cancel{2}x}{\cancel{2}\cdot 1} \quad \frac{\cancel{3}x^2}{\cancel{3}\cdot 2!}$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$\Rightarrow \sum \frac{x^n}{n!} = e^x$

$\frac{2^2 x^2}{\cancel{2!}} = 2\cdot x^2$

$\downarrow$

$4x = 2\cdot 2x$

Ex give a power series exp. for e^{2x}?

$\sum \frac{x^n}{n!} = e^x$

sub: $x = 2x$

$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots$

$\downarrow d/dx$ $\quad \frac{2^2 x^2}{2!} \rightarrow \frac{\cancel{2}^2 \cdot 2x}{\cancel{2}!} \qquad \frac{2^3 \cdot x^3}{3!} \rightarrow \frac{2^3 \cdot \cancel{3} x^2}{\cancel{3}!}$

$e^{2x} \rightarrow e^{2x}\cdot 2$

$= 2 + 2^2 x + 2^2 \cdot x^2$

$= 2 + 2x + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \dots = 2\overbrace{\left(1 + x + \frac{x^2}{2!} + \dots\right.}^{e^{2x}}$