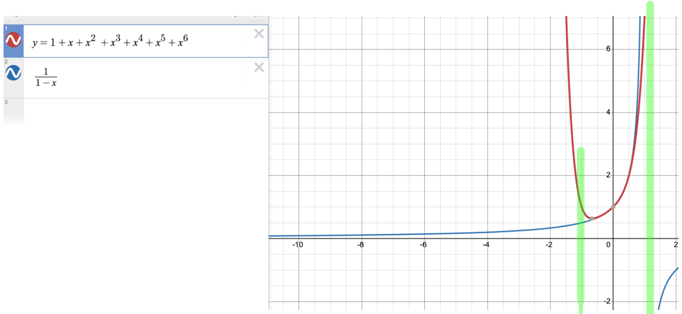
More Power series

When a series converges — we can manipulate both sides. (substitute x, derivative, integrate)



Recall

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{when } x \in (-1,1)$$

$$= 1 + x + x^2 + x^3 + x^4 + \dots$$

Substitute

sub $-x^2$ in for x in $\sum_{n=0}^{\infty} x^n$:

$$\sum_{n=0}^{\infty} (-x^2)^n = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \dots = \frac{1}{1-(-x^2)} \quad \text{when } -x^2 \in (-1,1)$$

$$\Updownarrow$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \dots \qquad = \frac{1}{1+x^2} \qquad |-x^2| < 1$$

$$-1 < -x^2 < 1$$

$$1 > x^2 > -1$$

square root

$$1 > |x|$$

Recap:

$$\boxed{f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots \quad \text{for } |x| < 1}$$

$$\frac{1}{1+(.1)^2} \approx 1 - \underset{\frac{1}{100}}{\underset{\|}{(.1)^2}} + \underset{\frac{1}{10000}}{\underset{\|}{(.1)^4}} \approx \overbrace{1 - \frac{1}{100}}^{.99} + \overbrace{\frac{1}{10{,}000}}^{+\,0.0001} \approx .9901$$

$$\left(\frac{1}{10}\right)^2 = \frac{1}{100} \qquad \left(\frac{1}{10}\right)^4 = \frac{1}{10000}$$

Note: $x = 5$, $|5| \not< 1 \Rightarrow \frac{1}{1+5^2} = \frac{1}{26}$ vs. $1 - 5^2 + 5^4 \approx 1 - 25 + 625 = 600$

Ex

Find a power series representation & it's interval of convergence

$f(x) = \dfrac{1}{2+x^2}$

Strategy: manipulate to make it similar to a known series

eg, $\dfrac{1}{1+x^2}$ or $\dfrac{1}{1-x}$

replace $-\dfrac{x^2}{2} = u$

① $\dfrac{1}{2+x^2} = \dfrac{1}{2\left(1+\frac{x^2}{2}\right)} = \dfrac{1}{2\left(1-\left(-\frac{x^2}{2}\right)\right)} = \dfrac{1}{2(1-u)} = \dfrac{1}{2}\left\{\dfrac{1}{1-u}\right\} = \dfrac{1}{2}\sum_{n=0}^{\infty} u^n$ w/ $|u|<1$

$= \underbrace{\dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{-x^2}{2}\right)^n}_{\text{power series rep.}}$ w/ $\left|\dfrac{-x^2}{2}\right|<1$

② get int. of conv. by solve $\left|\dfrac{-x^2}{2}\right| < 1$

$\shortparallel$

$\left|\dfrac{x^2}{2}\right| < 1$

$-1 < \dfrac{x^2}{2} < 1$

$\underbrace{-2 <}_{\text{always}} x^2 < 2$ (square root $|x| < \sqrt{2} \Rightarrow x \in (-\sqrt{2}, \sqrt{2})$

③ check endpoints $x = \sqrt{2}$

$\Rightarrow \dfrac{1}{2}\sum_{n=0}^{\infty}\left(-\dfrac{(\sqrt{2})^2}{2}\right)^n = \dfrac{1}{2}\sum\left(\dfrac{-2}{2}\right)^n = \dfrac{1}{2}\sum(-1)^n$ diverges (similarly for $x = -\sqrt{2}$)

Int. of Conv $= (-\sqrt{2}, \sqrt{2})$

Ex start w/

$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+x^5+\dots = \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1$$

$(1-x)^{-1}$

$$\Big\downarrow \frac{d}{dx}$$

$$-1(1-x)^{-2}(-1)$$

$$= \frac{1}{(1-x)^2} = 1+2x^{\boxed{1}}+3x^{\boxed{2}}+4x^{\boxed{3}}+5x^{\boxed{4}}+\dots = \sum_{n=0}^{\infty} (n+1)x^n \quad \text{for } |x|<1$$

Ex. we'll see:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \underbrace{\frac{x^0}{0!}}_{=1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$\Big\downarrow \frac{d}{dx}$$

$$= 0 + 1 + \frac{\cancel{2}x^1}{\underbrace{\cancel{2}!}_{=2\cdot 1}} + \frac{\cancel{3}x^2}{\cancel{3}\cdot 2!} + \frac{\cancel{4}x^3}{\cancel{4}\cdot 3!} + \frac{5x^4}{5!}$$

$$\frac{d}{dx}(e^x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = e^x$$