10.6 Power Series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

c ← center

Two Conventions: $0^0 = 1$ b/c: $2^0=1,\ 1^0=1,\ \frac{1}{2}^0=1,\ (\frac{1}{3})^0=1,\ (\frac{1}{4})^0=1$

$0! = 1$ b/c: $n! = n(n-1)!$

so $1! = 1 \cdot 0!$, and $1! = 1$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$0! \stackrel{?}{=} 0!$

Power Series for e^x (we'll see why later)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Find the x-values for which this series converges. (interval of convergence), eg (a,b), $[a,b)$, $(a,b]$, $[a,b]$, $\mathbb{R}$

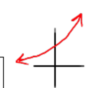
Use ratio test:

x = fixed

$$\left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right| \xrightarrow[n\to\infty]{\lim} 0 < 1 \Rightarrow \text{converges } \forall x.$$

for all

$$\boxed{\text{Interval of Convergence} = \mathbb{R}}$$



$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)\cdot 3^n}$$

Int. of Conv.

step ① Ratio Test:

$$\left| \frac{\frac{(x-2)^{n+1}}{(n+2)\cdot 3^{n+1}}}{\frac{(x-2)^n}{(n+1)\,3^n}} \right| = \left| \frac{\overset{(x-2)^n\cdot(x-2)}{(x-2)^{n+1}}}{(n+2)\,\underset{3^n\cdot 3}{3^{n+1}}} \cdot \frac{(n+1)\,3^n}{(x-2)^n} \right| \xrightarrow{\lim_{n\to\infty}} \left| \frac{(x-2)(n+1)}{(n+2)\cdot 3} \right| \xrightarrow{n\to\infty} \left| \frac{x-2}{3} \right| < 1$$

some formula involving x

step ② $\left|\frac{x-2}{3}\right| < 1 \Rightarrow -1 < \frac{x-2}{3} < 1$

mult. by 3

$-3 < x-2 < 3$

add 2

$-1 < x < 5$

interval notation

$x \in (-1, 5)$

step ③ B/C "$=1$" Ratio Test is inconclusive $\Rightarrow$ check endpoints explicitly

$x=5$ makes R.T. $=1$ so: sub $x=5$ and check convergence manually into OG.

$$\sum_{n=0}^{\infty} \frac{(5-2)^n}{(n+1)3^n} = \sum_{n=0}^{\infty} \frac{3^n}{(n+1)3^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

div. b/c $\approx \sum_{n=0}^{\infty} \frac{1}{n}$ div. $p=1$ series

'L.C.T.'

$(AB)^n = A^n \cdot B^n$

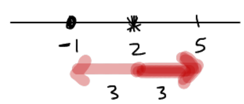
$x=-1$

$$\sum_{n=0}^{\infty} \left(\frac{(-1-2)^n}{(n+1)3^n}\right) = \sum_{n=0}^{\infty} \frac{(-3)^n}{(n+1)(3^n)} = \sum_{n=0}^{\infty} \frac{(-1)^n\cdot 3^n}{(n+1)3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

A.S.T. $\frac{1}{n+1}$ is decreasing $\Rightarrow$ converges

step ④ Int. of Convergence $[-1, 5)$

Midpoint $= \frac{5-1}{2} = \frac{4}{2} = 2$

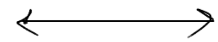
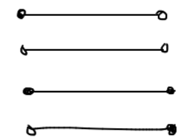


Radius of Convergence

$\frac{\text{dist}(-1,5)}{2} = \frac{5-(-1)}{2} = 3$

Recap

Every power series does <u>exactly</u> one of the following

1. Converge only for one point $x=a$, diverge elsewhere
2. Converge everywhere ($\mathbb{R}$) 
3. converge on some interval. 

When a series converges — you can

1. "do algebra" on both sides
2. take derivatives } both sides
3. integrate

Ex

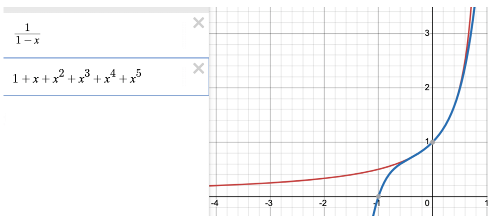
$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$$

geometric series
ratio of consecutive term is const

$$\frac{x^4}{x^3} = x$$

$$= \frac{1}{1-x}$$

interval of conv. $(-1,1)$



here $c=1$
$r=x$

$$\frac{x^3}{x^2} = x$$

$$\sum c r^n = \frac{c}{1-r}$$

when $|r|<1$