Maclaurin & Taylor polynomials & series

1. Find the fourth degree Maclaurin polynomial for the function $f(x) = \ln(x+1)$.

$f(x) = \ln(x+1)$	$f(0) = 0$
$f'(x) = \frac{1}{x+1}$	$f'(0) = 1$
$f''(x) = -\frac{1}{(x+1)^2}$	$f''(0) = -1$
$f^{(3)}(x) = \frac{2}{(x+1)^3}$	$f^{(3)}(0) = 2$
$f^{(4)}(x) = -\frac{6}{(x+1)^4}$	$f^{(4)}(0) = -6$

Use the above calculations to write the fourth degree Maclaurin polynomial for $\ln(x+1)$.

$$\begin{aligned} p_4(x) &= \frac{1}{0!}(0) + \frac{1}{1!}(1)x + \frac{1}{2!}(-1)x^2 + \frac{1}{3!}(2)x^3 + \frac{1}{4!}(-6)x^4 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \end{aligned}$$

Now write the Maclaurin *series* for $\ln(x+1)$.

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n$$

2. Find the fourth degree Taylor polynomial at $x = 1$ for the function $g(x) = \sqrt{x}$.

$g(x) = \sqrt{x}$	$g(1) = 1$
$g'(x) = \frac{1}{2}x^{-1/2}$	$g'(1) = \frac{1}{2}$
$g''(x) = -\frac{1}{4}x^{-3/2}$	$g''(1) = -\frac{1}{4}$
$g^{(3)}(x) = \frac{3}{8}x^{-5/2}$	$g^{(3)}(1) = \frac{3}{8}$
$g^{(4)}(x) = -\frac{15}{16}x^{-7/2}$	$g^{(4)}(1) = -\frac{15}{16}$

Use the above calculations to write the fourth degree Taylor polynomial at $x = 1$ for $\sqrt{x}$.

$$\begin{aligned} p_4(x) &= \frac{1}{0!}\cdot 1 + \frac{1}{1!}\cdot\frac{1}{2}(x-1) + \frac{1}{2!}\cdot\frac{-1}{4}(x-1)^2 + \frac{1}{3!}\cdot\frac{3}{8}(x-1)^3 + \frac{1}{4!}\cdot\frac{-15}{16}(x-1)^4 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 \end{aligned}$$

Here's the pattern for the full expansion:

$$1+\sum_{n=1}^{+\infty}(-1)^n\frac{1}{n!}\cdot\frac{(-1)(1)(3)\cdots(2n-3)}{2^n}(x-1)^n$$

3. Find the second degree Taylor polynomial at $x=2$ for the function $h(x)=x^2+3x-1$.

$$\begin{array}{ll} h(x)=x^2+3x-1 & h(2)=9 \\ h'(x)=2x+3 & h'(2)=7 \\ h''(x)=2 & h''(2)=2 \end{array}$$

$$h(x)=\frac{9}{0!}+\frac{7}{1!}(x-2)+\frac{2}{2!}(x-2)^2=(x-2)^2+7(x-2)+9$$

Note that all we have really done is "rearrange" $h(x)$ . . .

$$h(x)=(x-2)^2+7(x-2)+9=(x^2-4x+4)+7x-14+9=x^2+3x-1$$

4. Use your work from the front page to write the first, second, third, and fourth degree Taylor polynomials at $x=1$ for the function $g(x)=\sqrt{x}$.

$$p_1(x)=1+\frac{1}{2}(x-1)$$

$$p_2(x)=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2$$

$$p_3(x)=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3$$

$$p_4(x)=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3-\frac{5}{128}(x-1)^4$$

Now evaluate each of these polynomials at $x=1.21$, $x=1.96$, and $x=16$.

$p_1(1.21)=1.105$	$p_1(1.96)=1.48$	$p_1(16)=8.5$
$p_2(1.21)\simeq 1.099488$	$p_2(1.96)=1.3648$	$p_2(16)=-19.625$
$p_3(1.21)\simeq 1.100066$	$p_3(1.96)=1.420096$	$p_3(16)=191.3125$
$p_4(1.21)\simeq 1.099990$	$p_4(1.96)\simeq 1.386918$	$p_4(16)=-1786.22656$
$\sqrt{1.21}=1.1$	$\sqrt{1.96}=1.4$	$\sqrt{16}=4$