Interval of convergence

1. Find the interval of convergence for the following Taylor series:

$$\frac{1}{6-x} = \frac{1}{4} + \frac{1}{4^2}(x-2) + \frac{1}{4^3}(x-2)^2 + \cdots + \frac{1}{4^{n+1}}(x-2)^n + \cdots$$

$$= \sum_{n=0}^{+\infty} \frac{1}{4^{n+1}}(x-2)^n$$

"Apply" the ratio test . . . .

(i) Form the fraction $|a_{n+1}|/|a_n|$ and simplify as much as possible. (You need to include the power of x.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{|x-2|^{n+1}}{4^{(n+1)+1}}}{\frac{|x-2|^n}{4^{n+1}}} = \frac{|x-2|^n \cdot |x-2|}{4^n \cdot 4^2} \cdot \frac{4^n \cdot 4}{|x-2|^n} = \frac{|x-2|}{4}$$

(ii) Evaluate the limit:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-2|}{4} = \frac{|x-2|}{4}$$

(iii) Recalling that the limit in (ii) needs to be less than 1, find the *open* interval of convergence for the series, in the form $a < x < b$.

$$\frac{|x-2|}{4} < 1 \Longrightarrow |x-2| < 4 \Longrightarrow -4 < x-2 < 4 \Longrightarrow -2 < x < 6$$

(iv) Now "check the endpoints" of the open interval you found for convergence.

When $x = 6$, the series "in play" is

$$\frac{1}{4}+\frac{1}{16}(6-2)+\frac{1}{64}(6-2)^2+\frac{1}{256}(6-2)^3+\cdots = \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\cdots = +\infty$$

This series diverges (by Divergence Test) so 6 is not in the interval of convergence.

When $x = -2$, the series "in play" is

$$\frac{1}{4}+\frac{1}{16}(-2-2)+\frac{1}{64}(-2-2)^2+\frac{1}{256}(-2-2)^3+\cdots = \frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4}+\cdots = ??$$

This series diverges (by Divergence Test) so -2 is not in the interval of convergence.

Final answer:

$$\frac{1}{6-x} = \sum_{n=0}^{+\infty} \frac{1}{4^{n+1}}(x-2)^n \text{ for } x \in (-2, 6)$$

2. Find the interval of convergence for the Taylor series:

$$e^{-x/3} = 1 - \frac{1}{3}\,x + \frac{1}{2!3^2}\,x^2 + \cdots + \frac{(-1)^n}{n!3^n}\,x^n + \cdots$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!3^n} x^n$$

"Apply" the ratio test . . . .

(i) Form the fraction $|a_{n+1}|/|a_n|$ and simplify as much as possible. (You need to include the power of x.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{|x|^{n+1}}{(n+1)!3^{n+1}}}{\frac{|x|^n}{n!3^n}} = \frac{|x|^n \cdot |x|}{(n+1)n!3^n \cdot 3} \cdot \frac{n!3^n}{|x|^n} = \frac{|x|}{3(n+1)}$$

(ii) Evaluate the limit:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|}{3(n+1)} = \frac{|x|}{3}\left[\lim_{n\to\infty} \frac{1}{n+1}\right] = \frac{|x|}{3} \cdot 0 = 0$$

(iii) Recalling that the limit in (ii) needs to be less than 1, find the interval of convergence for the series.

$$e^{-x/3} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!3^n} x^n \text{ for } -\infty < x < +\infty$$