Interval of convergence

1. Find the interval of convergence for the following Maclaurin series:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n + \cdots$$
$$= \sum_{n=0}^{+\infty} \frac{1}{n!}x^n$$

"Apply" the ratio test . . . .

(i) Form the fraction $|a_{n+1}|/|a_n|$ and simplify as much as possible. (You need to include the power of x - and the absolute value signs.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}} = \frac{|x|^n \cdot |x|}{(n+1)n!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1}$$

(ii) Evaluate the limit:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = |x| \cdot \lim_{n\to\infty}\frac{1}{n+1} = |x| \cdot 0 = 0$$

(iii) Recalling that the limit in (ii) needs to be less than 1, find the *open* interval of convergence for the series, in the form $a < x < b$.

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1 \text{ for all values of } x \Longrightarrow$$

The inberval of convergence for the Maclaurin series for e^x is $(-\infty, +\infty)$.

2. Find the interval of convergence for the following Taylor series:

$$\frac{1}{4} + \frac{1}{4^2}(x-2) + \frac{1}{4^3}(x-2)^2 + \cdots + \frac{1}{4^{n+1}}(x-2)^n + \cdots$$
$$= \sum_{n=0}^{+\infty} \frac{1}{4^{n+1}}(x-2)^n$$

"Apply" the ratio test . . . .

(i) Form the fraction $|a_{n+1}|/|a_n|$ and simplify as much as possible. (You need to include the power of x.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{|x-2|^{n+1}}{4^{(n+1)+1}}}{\frac{|x-2|^n}{4^{n+1}}} = \frac{|x-2|^n \cdot |x-2|}{4^n \cdot 4^2} \cdot \frac{4^n \cdot 4}{|x-2|^n} = \frac{|x-2|}{4}$$

(ii) Evaluate the limit:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-2|}{4} = \frac{|x-2|}{4}$$

(iii) Recalling that the limit in (ii) needs to be less than 1, find the *open* interval of convergence for the series, in the form $a < x < b$.

$$\frac{|x-2|}{4} < 1 \Longrightarrow |x-2| < 4 \Longrightarrow -4 < x-2 < 4 \Longrightarrow -2 < x < 6$$

(iv) Now "check the endpoints" of the open interval you found for convergence.

When $x = 6$, the series "in play" is

$$\frac{1}{4}+\frac{1}{16}(6-2)+\frac{1}{64}(6-2)^2+\frac{1}{256}(6-2)^3+\cdots = \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\cdots = +\infty$$

This series diverges (by Divergence Test) so 6 is not in the interval of convergence.

When $x = -2$, the series "in play" is

$$\frac{1}{4}+\frac{1}{16}(-2-2)+\frac{1}{64}(-2-2)^2+\frac{1}{256}(-2-2)^3+\cdots = \frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4}+\cdots = ??$$

This series diverges (by Divergence Test) so -2 is not in the interval of convergence.

Final answer:

$$\frac{1}{6-x} = \sum_{n=0}^{+\infty} \frac{1}{4^{n+1}}(x-2)^n \text{ for } x \in (-2,6)$$