4. The Maclaurin series for the function $\sin x$ is shown below Carefully show that the interval of convergence for the series is $-\infty < x < +\infty$.

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Apply Ratio Test.
First find the ratio $|a_{n+1}|/|a_n|$.

$$\frac{\left|a_{n+1}\right|}{\left|a_n\right|} = \frac{\left|\frac{(-1)^{n+1}}{(2(n+1)+1)!} x^{2(n+1)+1}\right|}{\left|\frac{(-1)^n}{(2n+1)!} x^{2n+1}\right|} = \frac{(2n+1)!|x|^{2n+3}}{(2n+3)!|x|^{2n+1}}$$

$$\frac{\left|a_{n+1}\right|}{\left|a_n\right|} = \frac{(2n+1)!|x|^2}{(2n+3)(2n+2)(2n+1)!} = \frac{|x|^2}{(2n+3)(2n+2)}$$

Now take the limit . . . .

$$\lim_{n\to+\infty} \frac{\left|a_{n+1}\right|}{\left|a_n\right|} = \lim_{n\to+\infty} \frac{|x|^2}{(2n+3)(2n+2)} = |x|^2 \left[\lim_{n\to+\infty} \frac{1}{(2n+3)(2n+2)}\right]$$

$$\lim_{n\to+\infty} \frac{\left|a_{n+1}\right|}{\left|a_n\right|} = |x|^2 \cdot 0 = 0$$

Since this limit is less than 1 regardless of the value of x, the interval of convergence for the power series is $(-\infty, +\infty)$.

5. Find the fifth degree Taylor polynomial for the function $f(x) = \sin x + \cos x$.

Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ for } -\infty < x < +\infty$$

we can get (by differentiating)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n} \text{ for } -\infty < x < +\infty$$

The fifth degree Taylor polynomial for $\sin x$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ and the fifth degree Taylor polynomial for $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$.

Answer: The fifth degree Taylor poly.l (at 0) for $\sin x + \cos x$ is

$$1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$$

6. Use the fifth degree Taylor poly. from problem 5 to estimate $\sin 1 + \cos 1$.

$$\sin 1 + \cos 1 \simeq 1 + 1 - \frac{1}{2} - \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{83}{60}$$

7. Use a sixth degree Taylor polynomial to estimate

$$\int_0^1 \sin(x^2)\, dx$$

We have from above that

$$\sin \square = \square - \frac{1}{3!}\square^3 + \frac{1}{5!}\square^5 + \cdots$$

So

$$\sin x^2 = (x^2) - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 + \cdots = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} + \cdots$$

We are asked to use a sixth degree polynomial, and $\sin x^2 \simeq x^2 - \frac{1}{6}x^6$.

Answer:

$$\int_0^1 \sin(x^2)\, dx \simeq \int_0^1 x^2 - \frac{1}{6}x^6\, dx = \left[\frac{1}{3}x^3 - \frac{1}{42}x^7\right]_0^1 = \frac{1}{3} - \frac{1}{42} = \frac{13}{42}$$

8. Find the Maclaruin series for the funciton $\tan^{-1} x$. (Derive it - either from Taylor formula (not recommended) or some other method. For example, the power series for the function $1/(1-x)$ might be helpful.)

Then show that the interval of convergence for the series is $[-1, 1]$.

From earlier in Chapter 11, we know that

$$\frac{1}{1-\Box} = 1 + \Box + \Box^2 + \Box^3 + \cdots$$

$$\implies \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\int \frac{1}{1+x^2}\, dx = \int 1 - x^2 + x^4 - x^6 + \cdots\, dx$$

$$\tan^{-1} x = C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots$$

$$\text{If } x = 0 \implies \tan^{-1}(0) = C \implies C = 0$$

So the Maclaurin series for $\tan^{-1} x$ is

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

Now for the interval of convergence . . . .

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{2n+1}{2n+3}|x|^2 = |x|^2$$

By the Ratio Test, we know that the Maclaurin series for $\tan^{-1} x$ converges for all x such that $|x|^2 < 1$. That is, we know the series converges whenever $-1 < x < 1$.

We also know the series diverges if $x < -1$ or if $x > 1$. We need to check $x = -1$ and $x = 1$.

When $x = 1$, the resulting series is

$$(1) - \frac{1}{3}(1)^3 + \frac{1}{5}(1)^5 - \cdots = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}$$

This series converges by the Alternating Series Test.

When $x = -1$, the resulting series is

$$(-1) - \frac{1}{3}(-1)^3 + \frac{1}{5}(-1)^5 - \cdots = -1 + \frac{1}{3} - \frac{1}{5} + \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{2n+1}$$

This series converges by the Alternating Series Test.

Final Answer:

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots = \sum \frac{(-1)^n}{2n+1} x^{2n+1} \text{ for } x \in [-1, 1]$$