MA163 Fri. Wk 9

Exam 3: week from Mon.

Today: Taylor's Remainder theorem.

Next week: Mon, Wed, Thur, } Arc Length, Parametric Curves

Fri, → Exam 3 Review

Study Guide: Monday

Taylor's Remainder Thm:

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

$f(x) = 1 + 3x^2$

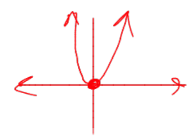
$f(0)$

$f'(x) = 6x$
$f'(0) = 0$

1st Degree ($n=1$) Taylor Poly for $f(x)$ @ $x=0$ (Maclaurin)

$$f(x) = \underbrace{\frac{f^{0}(0)(x-0)^0}{0!}}_{n=0} + \underbrace{\frac{f'(0)(x-0)^1}{1!}}_{n=1} = 1 + 0\frac{(x-0)^1}{1} = 1$$

For the first degree Taylor poly, the 2nd derivative controls the error (remainder)



Taylor's Inequality (Remainder Theorem)

If $f(x)$ = Taylor Polynomial (∞)

$T_n(x)$ = n^{th} degree Taylor approx

$$R_n(x) = |f(x) - T_n(x)| \leq f^{(n+1)}(u)\frac{|x-a|^{n+1}}{(n+1)!}$$

where u is the x-value that maximizes the $(n+1)$-derivative

useful way to state Tay. Rem Thm

$$R_n(x) \leq \frac{M \cdot |x-a|^{n+1}}{(n+1)!}$$

where $M \geq f^{(n+1)}(x)$ where x is close to a $|x-a| < d$, Ex: set $f(x) = e^x$ on $[-2,2]$. Approximate $e^{1.5}$ with error less than 10^{-3}.

Since 1.5 is close-ish to 0 we can use Maclaurin ($a=0$)

$f(x) = e^x$ $\quad f(0) = 1$

$f'(x) = e^x$ $\quad f'(0) = 1$

$f''(x) = e^x$ $\quad f''(0) = 1$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

How many terms are needed to approximate $e^{1.5}$

Taylor Rem thm $\quad |R_n(x)| \le \dfrac{M x^n}{n!}$

set $x = 1.5$

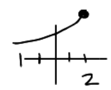
choose $M \geq f^{(n+1)}(x)$ where $x \in [-2,2]$

i.e., what number is bigger than the (n+1)-th derivative of e^x on [-2,2]

$$\underbrace{\text{(n+1)-th derivative}}_{e^x}$$

$$\boxed{R_n(1.5) \le \frac{9(1.5)^n}{n!}}$$

the error @ $x = 1.5$



$e < 3$

$e^x < 3^x$ on 3^x when $x = 2$

$= 9$

$$\boxed{M = 9 > e^x \text{ when } x \in [-2,2]}$$

Solve for n $\quad \dfrac{9(1.5)^n}{n!} < 10^{-3}$

$10^3 \cdot 9 (1.5)^n < n!$

$9000 (1.5)^n < n!$ $\quad \Rightarrow \quad n = 10$ works

$$\boxed{e^{1.5} \approx \sum_{n=0}^{10} \frac{(1.5)^n}{n!}}$$

and this is within 10^{-3} of actual value.

Instead of Macl, do taylor @ $a = 1$ b/c 1 is closer to 1.5

$$\sum \frac{f^{(n)}(1)(x-1)^n}{n!}$$

$$R_n(1.5) \le \frac{M|\overset{1.5}{x}-1|^{n+1}}{(n+1)!}$$

$f^{n+1}(x) = e^x < 3^x < 9.$ on $[-2,2]$

$$\frac{9(0.5)^{n+1}}{(n+1)!} = \frac{9(1.5-1)^{n+1}}{(n+1)!} < 10^{-3}$$

$$9000(0.5)^{n+1} < (n+1)!$$