Monday – Week 9

Exams Returned Wed / Thur

polynomials are the building blocks of all functions

Ex $f(x) = e^x$
$g(x) = \sin(x)$ ——— each equals an infinite polynomial
$h(x) = \tan^{-1}(x)$

Power Series: (infinite polynomial)

eg. $3 + 5x^3 + 15x^4 + 7x^5$ is a poly

$= \sum_{n=0}^{5} c_n x^n$ where $c_0 = 3,\ c_1 = 0,\ c_2 = 0$
$c_3 = 5,\ c_4 = 15,\ c_5 = 7$

$$\sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Def'n Power Series centered at $x = 0$: $\sum_{n=0}^{+\infty} c_n x^n$

centered $x = a$ is

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots$$

Two Conventions: $0^0 = 1$

$4^0 = 1,\ 3^0 = 1,\ 2^0 = 1,\ 1^0 = 1, \dots 0^0 = 1$

$0! = 1$ b/c

$n! = n(n-1)!$

& if $0! = 0$ then

$1! = 1(0!) = 0$

Ex $\sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5140}$

Note: x is a variable

Question: Does this series converge to some # or diverge?
(For what values of x does it converge / diverge?)

To answer this: ———

Ratio Test: $\rho < 1$ converge | $\rho > 1$ diverge | $\rho = 1$??? (we'll test case by case)

$$\left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \left| \frac{\cancel{x^{n+1}}}{(n+1)!} \cdot \frac{n!}{\cancel{x^n}} \right| = \left| \frac{x}{n+1} \right| \xrightarrow[n \to \infty]{} \left| \frac{x}{\infty} \right| = 0 = \rho$$

Ex 2 $\displaystyle\sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}$ series from Ex 1 centered at $x=a$

Determine for what values of x the series converges

Step 1: Ratio Test.

$$\left|\frac{(x-a)^{n+1}}{(n+1)!}\cdot\frac{n!}{(x-a)^n}\right| = \left|\frac{(x-a)}{(n+1)}\right| \longrightarrow 0$$

Converges for all x. b/c limit < 1.

Ex 3 $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$ when converge?

Step 1 Ratio Test.

$$\left|\frac{\dfrac{(x-2)^{n+1}}{(n+2)\,3^{n+1}}}{\dfrac{(x-2)^n}{(n+1)\,3^n}}\right| = \left|\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}\cdot\frac{(n+1)\,3^n}{(x-2)^n}\right| = \left|\frac{(x-2)(n+1)}{(n+2)\,3}\right| \xrightarrow[n\to\infty]{} \left|\frac{x-2}{3}\right| = \rho$$

Step 2 Converge: $\rho < 1$ so $\left|\dfrac{x-2}{3}\right| < 1$ or $-1 < \dfrac{x-2}{3} < 1$ $\Big\}$ $-3 < x-2 < 3$, $-1 < x < 5$ so $\Big\}$ iff $x \in (-1, 5)$ series converges

$\rho > 1 \Rightarrow$ diverge

$\rho = 1 \Rightarrow$??? $\left|\dfrac{x-2}{3}\right| = 1 \Rightarrow$ $\dfrac{x-2}{3} = 1$ or $\dfrac{x-2}{3} = -1$

$x-2=3$, $x=5$ $x-2=-3$, $x=-1$

Step 3 Look at series given by the endpoints of our interval $(-1 \,\&\, 5)$

$x=-1$ $\xrightarrow{\text{substitute}}$ $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{(n+1)3^n} = \sum_{n=0}^{\infty}\frac{(-3)^n}{(n+1)3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot\cancel{3^n}}{(n+1)\cancel{3^n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$

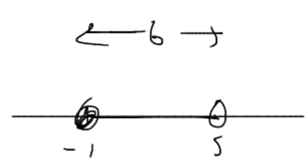
A.S.T. $\Rightarrow$ converges $\left(\displaystyle\lim_{n\to\infty}\frac{1}{n+1} = 0\right)$ ... $x=-1$ works

$x=5$ $\xrightarrow{\text{substitute}}$ $\displaystyle\sum_{n=0}^{\infty}\frac{(5-2)^n}{(n+1)3^n} = \sum_{n=0}^{\infty}\frac{3^n}{(n+1)3^n} = \underbrace{\sum_{n=0}^{\infty}\frac{1}{n+1}}_{\text{Harmonic}}$ diverges | $x=5$ doesn't work

Step 4. Ans: $[-1, 5)$

Called: Interval of Convergence

Radius of Conv. $= 3$



$R = \text{radius} = 3$

Every Power Series does <u>one</u> of three things ———

1. Converges only for $x = a$, diverges elsewhere.

2. Converges everywhere (for all $x \in \mathbb{R}$)

3. Converges on an interval.

R = radius of this interval (3 in this case)

R = radius of convergence