Week 9 — Mon

Start: MacLaurin Series for $\sin(x)$, $\cos(x)$, e^x — then relate them all w/ π, i

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Int. of Conv. $= \mathbb{R}$

See: HW. 10.6–10.8 part 4

$$\sum_{n=0}^{\infty} \frac{(\ln(6))^n}{n!} = e^{\ln(6)} = 6$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^{2n} \frac{x^{2n}}{(2n)!}$$

n		$f^{(n)}(0)$
0	$f = \sin x$	0
1	$f' = \cos x$	1
2	$f'' = -\sin x$	0
3	$f''' = -\cos x$	-1

Application / Example:

In physics, machine learning! we extend Taylor Series (MacLaurin) to complex #'s. $(i = \sqrt{-1})$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \dots$$

$n=0$ $\quad n=1$

$i^2 = (\sqrt{-1})^2 = -1$

$(i\theta)^2 = i^2\theta^2 = -\theta^2$

$(i\theta)^4 = i^4\theta^4 = (i^2)^2\theta^4 = \theta^4$

$(i\theta)^6 = -\theta^6$

$\vdots$

permute

$$= 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + \dots + i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^7}{7!} + \dots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots + i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \dots$$

$(i\theta)^3 = i^2 \cdot i\theta^3 = -i\theta^3$

$(i\theta)^5 = i^5\theta^5 = i^4 \cdot i\theta^5 = i\theta^5$

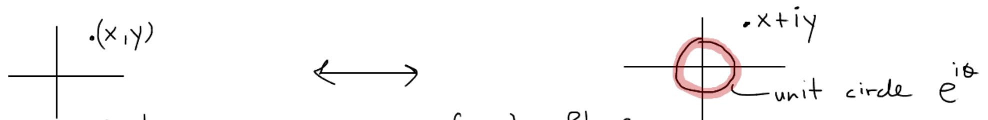
$(i\theta)^7 = i^5 \cdot i^2\theta^7 = i(-1)\theta^7 = -i\theta^7$

$$= \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots}_{} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)}_{}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$ ← Euler's Identity

Evaluate @ $\theta = \pi \Rightarrow$ $e^{i\pi} = \cos(\pi) + i \cdot \sin(\pi) = -1 + i \cdot 0 = -1$ $\quad e^{i\pi} = -1 \rightsquigarrow \boxed{e^{i\pi} + 1 = 0}$

Further Connections:



Cartesian Plane $\longleftrightarrow$ Complex Plane

Binomial Power Series:

(How to approximate $(1+x)^{-5/3}$, see HW part 1)

main idea:

For integer exponents, $(1+x)^2, (1+x)^3, (1+x)^4, \ldots$ we use as coeffs

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

level

0 1 2 3 4 → k

binomial coefs:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{24}{2\cdot 2} = 6$$

$\leftarrow$ level

$$\binom{4}{0} = 1$$

$$\frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)(n-2)\cdots\cancel{(n-k)!}}{k!\cancel{(n-k)!}}$$

$$= \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!}$$

coef

of ways to choose k items from a
bag of n items