Thur wk 9

hw Exercises

1. observe that evaluating the limit immediately gives 0/0
2. plan: expand tan & cos into sums (keep summation notation)
3. hint: pull out first few terms to avoid division by 0 upon evaluating limit

Part 4 #5

$K\sum x^n = \sum Kx^n$

$\sum_{n=0}^{\infty} f + \sum_{n=0}^{\infty} g = \sum_{n=0}^{\infty} f+g$

$$\lim_{x\to 0} \frac{\tan^{-1}(7x) - 7x\cos(7x) - \frac{343}{6}x^3}{x^5}$$

$$\tan^{-1}(7x) = 7x - \frac{(7x)^3}{3} + \frac{(7x)^5}{5} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{(7x)^{(2n+1)}}{2n+1}$$

$$7x\cos(7x) = 7x\left(1 - \frac{(7x)^2}{2!} + \frac{(7x)^4}{4!} - \dots\right) = 7x\sum_{n=0}^{\infty}(-1)^n\frac{(7x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(7x)^{2n+1}}{(2n)!}$$

$$\tan^{-1}(7x) - 7x\cos(7x) = \sum_{n=0}^{\infty}(-1)^n\frac{(7x)^{(2n+1)}}{2n+1} - \sum_{n=0}^{\infty}(-1)^n\frac{(7x)^{2n+1}}{(2n)!}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(7x)^{2n+1}}{2n+1} - \frac{(-1)^n(7x)^{2n+1}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n(7x)^{2n+1}\left(\frac{1}{2n+1} - \frac{1}{(2n)!}\right)$$

$\frac{2}{6} - \frac{3}{6} = \frac{-1}{6}$ $\qquad$ $\frac{4!-5}{5!} = \frac{19}{120}$

write out 1st few terms

$$= (-1)^0(7x)^1\left(\underbrace{\frac{1}{1} - \frac{1}{0!}}_{1-1=0}\right) \quad (-1)(7x)^3\left(\frac{1}{3} - \frac{1}{2}\right) \quad (-1)^2(7x)^5\left(\underbrace{\frac{1}{5} - \frac{1}{4!}}\right)$$

$n=0$: 0 $+$ $n=1$: $\frac{7^3x^3}{6}$ $+$ $n=2$: $7^5\left(\frac{19}{120}\right) = \frac{319333}{120}x^5$ $\qquad$ $n=3+$

$\frac{A-B}{C} = \frac{A}{C} - \frac{B}{C}$

given

$$= \lim_{x\to 0}\frac{\cancel{\frac{7^3x^3}{6}} + \frac{319333x^5}{120} + \sum_{n=3}^{\infty}(-1)^n(7x)^{2n+1}\left(\frac{1}{2n+1} - \frac{1}{(2n)!}\right) - \cancel{\frac{343}{6}x^3}}{x^5}$$

$\leftarrow$ cancel ✶

$7^{2n+1}\cdot x^{2n+1}$

$K\sum_n f(x) = \sum_n K\cdot f(x)$

$$= \lim_{x\to 0} \frac{319333}{120} + x^{-5}\sum_{n=3}^{\infty}(-1)^n(7x)^{2n+1}\left(\frac{1}{2n+1} - \frac{1}{(2n)!}\right)$$

$$= \lim_{x\to 0} \frac{319333}{120} + \sum_{n=3}^{\infty}(-1)^n 7^{2n+1}\underbrace{x^{2n+1}\cdot x^{-5}}_{x^{2n-4}}\left(\frac{1}{2n+1} - \frac{1}{(2n)!}\right)$$

$$= \lim_{x\to 0} \frac{319333}{120} + \sum_{n=3}^{\infty}(-1)^n 7^{2n+1} x^{2n-4}\left(\frac{1}{2n+1} - \frac{1}{(2n)!}\right)$$

$$= \frac{319333}{120} + \underbrace{\lim_{x\to 0}\sum \text{mm}}_{=0}$$

$\uparrow$ answer.

$n=3$ $\quad x^{6-4} = x^2 \longrightarrow 0$

$n=4$ $\quad x^{2\cdot 4-4} = x^4 \longrightarrow 0$

$n=$

Taylor's Remainder theorem

Suppose $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-a)^n}{n-1}$$

what can we say (estimate error) when we approximate this w/ a finite sum —

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}}_{\text{by using this as approximation}} + \frac{f''''(a)(x-a)^4}{4!} + \ldots$$

this term* controls error

$$E_n(x) = |f(x) - T_n(x)| \leq \frac{M \cdot |x-a|^n}{n!}$$ where $M \geq f^{(n)}(x)$

↑ constant

Ex.

How many terms are necessary to approximate sin(1) if using a MacL. series ______ w/in 10^{-2} error.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\text{Error} \le \frac{M \cdot x^{n+1}}{(n+1)!} \qquad M = 1 \text{ since } f(x) = \sin(x)$$

$$f^{(n)}(x) \le 1$$

solve $\dfrac{x^{n+1}}{(n+1)!} < 10^{-2}$ for $x = 1$

$$\frac{1^{n+1}}{(n+1)!} < 10^{-2} \Rightarrow 10^2 < (n+1)!$$

$\underline{\underline{n = 4}} = (n+1)! = 5! = 120$

$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$ @ $x = 1$ is w/in $\frac{1}{100}$ of true $\underline{\text{val}}$