Wed – Week 9

warm-up: what is a power series?

Ans: Infinite Polynomial: $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$

Today: we'll see: limits / derivatives / integrals of power series behave just as they do for finite polynomials

Monday: We saw functions can be represented (locally) by power series

$\downarrow$ interval of convergence

Ex 1: $f(x) = \frac{1}{1-x} = \frac{a}{1-r}$ w/ $a=1$, $r=x$ (geometric)

when $|r| = |x| < 1$

$= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} x^n$

So

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{when } |x| < 1}$$

We know

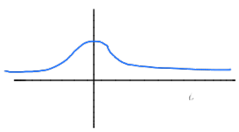
$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad \text{when } |x|<1}$$

we can ~~modify~~ this to represent other functions

relationship ?

substitute $-x^2$ as x in the box above)

Ex $f(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$



$$= \sum_{n=0}^{\infty} (-x^2)^n = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots$$

valid for $|-x^2| < 1$

$\Leftrightarrow$

$x^2 < 1$

$|x| < \sqrt{1} = 1$

always true

$-1 < x^2 < 1$ $\longrightarrow$ $x^2 - 1 < 0$

$(x-1)(x+1) < 0$

Interval of Convergence

$x \in (-1, 1)$

Radius of Convergence $= 1$

Test

False True False

-1 1

End Points

$x=-1 \Longrightarrow \sum_{n=0}^{\infty} (-(-1)^2)^n = 1 - 1 + 1 - 1 + 1 \ldots$ diverges (don't include $x=-1$

$x=1$ is similar

Ex $f(x) = \dfrac{1}{2+x^2} = \dfrac{1}{2(1+\frac{1}{2}x^2)} = \dfrac{1}{2(1-(-\frac{1}{2}x^2))}$ $\qquad \dfrac{1}{1-x}$

$a = \frac{1}{2}$

$r = -\frac{1}{2}x^2$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{1}{2}x^2\right)^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{1}{2}\right)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{1}{2}\right)^{n+1} \cdot x^{2n}$$ (*)

$(-1)^n \left(\frac{1}{2}\right)^n$

Interval of Convergence:

$|r| < 1$

$\|$

$\left|-\frac{1}{2}x^2\right| < 1$

or $|x^2| < 2$

$x^2 < 2$

$|x| < \sqrt{2}$

$$\boxed{(-\sqrt{2}, \sqrt{2})}$$

Endpoints: $x = \sqrt{2}$

plug into my new series (*)

Same for $x = -\sqrt{2}$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^{n+1} \cdot (\sqrt{2})^{2n}$$

$$= \sum (-1)^n \left(\frac{1}{2}\right)^{n+1} 2^n$$

$\left(\frac{1}{2} \cdot 2\right)^n = 1^n = 1$ (L)

$$= \sum (-1)^n \left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{2}\right) \cdot 2^n$$

$$= \sum (-1)^n \left(\frac{1}{2}\right)$$ diverges by A.S.T

In addition to substitution we have 3 other ways of modifying series to get new series

Properties of Power Series

1. limits: limits pass across sums:

$$\lim_{x \to x_1} \sum c_n x^n = \sum c_n x_1^n$$

2. Derivatives: Differentiate term by term

$$\frac{d}{dx} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} c_n x^n$$

3. Integrals:

$$\int \sum_{n=0}^{\infty} c_n x^n \, dx = \sum_{n=0}^{\infty} \int c_n x^n \, dx$$

Ex

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{when } |x| < 1}$$

$$\Big\downarrow \, d/dx$$

$$(1-x)^{-1} \longrightarrow -1(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = \sum_{n=0}^{\infty} (n+1) x^n$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \dots\right)$$

Fact ________________________________

why?

Start

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots$$

$$\Big\downarrow \text{ integrate}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11}$$

$= \sum (-1) \frac{1}{2n+1}$

conv. by AST

$\Rightarrow$ $x=1$ works

sub $x=1$

$$\tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

$$= \Big\downarrow$$

$$\frac{\pi}{4}$$

Wed. Week 9

Friday: (video-guided practice day)
NO synchronous class.

warm-up: what is a power series?
: infinite polynomial
: $\sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$

Monday: We can represent functions (locally) by power series.
(interval of convergence)

Ex 1 $f(x) = \frac{1}{1-x} \overset{\substack{\text{set} \\ a=1 \\ r=x}}{=\!=\!=} \frac{a}{1-r} = 1 + r + r^2 + r^3 + \ldots$

geometric series sum w/ $|r|<1$
$r \in (-1,1)$

So $\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{+\infty} x^n}$ for $x \in (-1,1)$

substitute $(-x^2)$ for x above

Ex $f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$

$= 1 - x^2 + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \ldots$

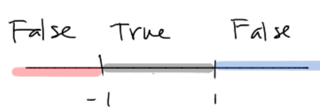
$= 1 - x^2 + x^4 - x^6 + x^8 - \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ ($*$)

Determine Interval of Convergence:

solve $|-x^2| < 1$
$|x^2| < 1$
$-1 < x^2 < 1$

$-1 < x^2$ always true

$x^2 < 1$
$\Updownarrow$
$x^2 - 1 < 0$
$(x-1)(x+1) < 0$



$\boxed{(-1,1)}$

Boundary Pts: $x = 1$, substitute into ($*$) $= 1 - 1 + 1 - 1 + 1 - \ldots$ diverges
$x = -1$ same

Ex $\boxed{f(x) = \dfrac{1}{2+x^2}} = \dfrac{1}{2(1-(-\frac{1}{2}x^2))} = \dfrac{1}{2} \cdot \dfrac{1}{1-(-\frac{1}{2}x^2)}$

Mimic $\dfrac{1}{1-x} = 1+x+x^2+x^3+\ldots$

substitute $-\frac{1}{2}x^2$ in for x in geometric series

$a = \frac{1}{2}$, $r = -\frac{1}{2}x^2$

$$\frac{\frac{1}{2}}{1-(-\frac{1}{2}x^2)} = \frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{+\infty} \frac{1}{2}\left(-\frac{1}{2}x^2\right)^n$$

$$= \sum_{n=0}^{+\infty} \frac{1}{2}\left(-\frac{1}{2}\right)^n x^{2n}$$

$$= \boxed{\sum_{n=0}^{+\infty} (-1)^n \left(\frac{1}{2}\right)^{n+1} x^{2n}}$$

Interval of Convergence: when is this representation valid

$|r|<1$ So $\left|\frac{-1}{2}x^2\right| < 1$

$x^2 < 2$

$\boxed{x \in (-\sqrt{2}, \sqrt{2})}$

as before the endpoints

$x = \sqrt{2} \longrightarrow \sum (-1)^n \left(\frac{1}{2}\right)^{n+1} (\sqrt{2})^{2n}$

$(\sqrt{2})^{2n} = (2^{\frac{1}{2}})^{2n} = ((2^{\frac{1}{2}})^2)^n = (2^{2/2})^n = (2^1)^n = 2^n$

$\left(\frac{1}{2}\right)^{n+1} = \left(\frac{1}{2}\right)^n \cdot \frac{1}{2}$

product

$(2^n)\left(\frac{1}{2}\right)^n \cdot \frac{1}{2} = \frac{1}{2}$

$\left(\frac{2}{2}\right)^n \cdot \frac{1}{2}$

$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2}$ diverges

$\Rightarrow$ exclude $x = \sqrt{2}$

same for $x = -\sqrt{2}$

Two other common ways to modify series to produce other (new) series

Theorem:

1. $\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx} c_n x^n$ (we can differentiate term by term)

2. $\int \sum_{n=0}^{\infty} c_n x^n \, dx = \sum_{n=0}^{\infty} \int c_n x^n \, dx$ (same for finite polynomials)

Ex

$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$

$\big\downarrow \, d/dx$

$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \ldots$

$\ldots + (n+1)x^n$

$\frac{1}{1-x} = (1-x)^{-1}$

d/dx

$-1(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$

$= \sum_{n=0}^{\infty} (n+1)x^n$

Int of Conv

$x \in (-1,1)$

Ex Given:

$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots$

$\big\downarrow \int dx$

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11}$

$\tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$

$\parallel$

$\frac{\pi}{4}$

$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right)$