Objective: We will use the concept of a *basis* to define the *dimension* of a subspace

BASIS FOR SUBSPACES

The xy-plane in $\mathbb{R}^3$ is a subspace consisting of all vectors in $\mathbb{R}^3$ with z coordinate equal to 0. Notice that

$$S = \{(1,0,0),(0,1,0),(1,1,0)\}$$

spans the xy-plane.

recall: the span of $\{\bar{v}_i\}$ is all possible linear combos of $\bar{v}_1, \bar{v}_2, \bar{v}_3, \ldots$

$$(6,2,0) = 6(1,0,0) + 2(0,1,0) + 0(1,1,0)$$

redundant:

notice: S is linearly dependent!

Also notice that if we throw out any one of the vectors in S and get a new set S', the new set still spans the xy-plane.

$S' = \{(1,0,0),(0,1,0)\}$

$S'' = \{(0,1,0),(1,1,0)\}$

$(6,2,0) = -4(0,1,0) + 6(1,1,0)$

S', S'' still span x-y plane.
both linearly independent

Fact: S', S'' are bases for the x-y plane subspace of $\mathbb{R}^3$.

1. Here S spans the xy-plane but is linearly dependent.
2. S' also spans the xy-plane but is linearly independent.
3. Sets like S' are preferred because they are "the most efficient way to describe every element in a subspace".

Definition. *A **basis** for a subspace V is a linearly independent spanning* ~~set~~ subset *of V.*

Example. *Find a basis of*

(a) The x-axis in $\mathbb{R}^3$. $= \{(x,0,0) \mid x \in \mathbb{R}\}$.

one basis is : $\{(1,0,0)\}$.

Not a basis $\Rightarrow$: $\{(1,1,0),(0,1,0)\}$

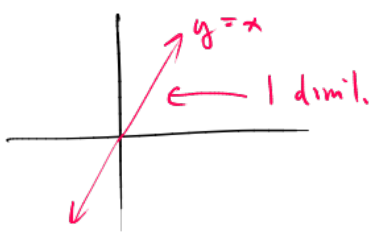
doesn't live in x-axis.

$\{(\pi,0,0)\}$ is a basis

$\frac{x}{\pi}(\pi,0,0) = (x,0,0)$

$\{(-1,0,0)\}$ is a basis

(b) The line through (0,0) in $\mathbb{R}^2$ intersecting the x-axis at an angle of $45°$.



$\bar{x} = \left\{ \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}, \ x_1 \in \mathbb{R} \right\}$

$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ = basis.

(c) The yz-plane in $\mathbb{R}^3$.

$= \{(0,x,y) \mid x,y \in \mathbb{R}\}$ 2-dim'l

$\{(0,1,0),(0,0,1)\}$ basis.

The standard unit vectors

$$(1,0) \text{ and } (0,1) \in \mathbb{R}^2$$

and

$$(1,0,0),(0,1,0),(0,0,1) \in \mathbb{R}^3$$

are the **standard basis vectors** for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. We sometimes call them $(e_1, e_2, \ldots, e_n)$.

Theorem (Row Space Basis)**.** *The nonzero row vectors of a matrix M in RREF form a basis for the row space of M.*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

row space: all of $\mathbb{R}^3$
basis: e_1, e_2, e_3

row space = x-y plane in $\mathbb{R}^3$
basis: $(1,0,0)$, $(0,2,0)$

1. Find a basis for the row space of $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix}$ $\begin{matrix} R_2 \\ -R_1 \\ R_2 - R_3 \end{matrix}$

Find basis for span of $(0,-1,2), (1,2,3), (1,1,5)$

$$\to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

basis is $\{(1,2,3), (0,1,-2)\}$

Theorem (Every subspace has a basis)**.** *Although a given subspace of $\mathbb{R}^n$ can have many different bases, the number of vectors in a basis does not change. In other words, if S is a fixed subspace of $\mathbb{R}^n$ any basis for S will have the same number of vectors. The* ***dimension*** *of a subspace is the number of vectors in a basis for the subspace.*

2. What is the dimension of the row space of the matrix above?

2

3. Give two different bases for the xz plane in $\mathbb{R}^3$.

4. Give a basis for the span of the following set of vectors.

$$S = \{(3,-3,5)\ ,\ (-2,-3,-3)\ ,\ (5,-15,9)\}.$$

What is the dimension of the span(S)?

$$\begin{bmatrix} 3 & -3 & 5 \\ -2 & -3 & -3 \\ 5 & -15 & 9 \end{bmatrix} \quad \begin{array}{c} \begin{array}{ccc} -5 & 30 & -10 \end{array} \\ \begin{bmatrix} 1 & -6 & 2 \\ -2 & -3 & -3 \\ 5 & -15 & 9 \end{bmatrix} \end{array} \quad \begin{array}{l} R_1+R_2 \\ R_2 \\ R_3 \end{array}$$

$$\begin{bmatrix} 1 & -6 & 2 \\ 0 & -15 & 1 \\ 0 & 15 & -1 \end{bmatrix} \begin{array}{l} 2R_1+R_2 \\ -5R_1+R_3 \end{array} \rightarrow \begin{bmatrix} 1 & -6 & 2 \\ 0 & 15 & -1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \quad R_2+R_3$$

ok to use these non-zero rows as basis

$$\begin{bmatrix} 1 & -6 & 2 \\ 0 & 1 & -1/15 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis: $\{(1,-6,2),\ (0,1,-1/15)\}$

$\dim = 2 = \dim(\text{span}(S))$

Example. *The orthogonal complement $a^{\perp}$ of the vector $a = (1,2,3)$ is the set of vectors $\vec{\mathbf{x}} = (x_1, x_2, x_3)$ of the form*

$$1x_1 + 2x_2 + 3x_3 = 0.$$

What is the dimension of (the Hyperplane) $a^{\perp}$?

Theorem (2). *Let S be a finite set of vectors in a nonzero subspace V of $\mathbb{R}^n$.*

(a) If S spans V but is not linearly dependent you can remove certain vectors from S to obtain a basis for V.

(b) If S is linearly independent but does not span you can add certain vectors to S to obtain a basis for V.

Example. *Apply the theorem above to the following two sets of vectors in the subspace $\mathbb{R}^3$ of $\mathbb{R}^3$.*

(a) $V = \{(1,0,1),(0,1,0)\}$

$\text{span}(V) \neq \mathbb{R}^3$.

$\text{span}\left(\{(1,0,1),(0,1,0),(0,0,1)\}\right) = \mathbb{R}^3$

$\{(1,0,1),(0,1,1),(0,0,1)\}$

(b) $W = \{(1,0,1),(0,1,0),(0,1,1),(1,1,0)\}$

$\text{span}(W) = \mathbb{R}^3$

if I remove any 1, the new set still span $\mathbb{R}^3$

Theorem (General statements about bases)**.** *(a) Every vector is a basis for some subspace.*

$\{(0,0)\}$ is a basis for $\vec{0}$. ← trivial subspace

$(1,-1,4)$ is a basis for $W = \{t(1,-1,4)\}$ $t \in \mathbb{R}$

(b) A set of k linearly independent vectors in a nonzero k-dimensional subspace is a basis.

(c) A set of k vectors that span a nonzero k-dimensional subspace of $\mathbb{R}^n$ must be linearly independent, hence is a basis.

(d) A set of fewer than k vectors in a nonzero k-dimensional subspace cannot span the subspace.

(e) A set with more than k vectors in a nonzero k-dimensional subspace is linearly dependent.

(f) If $B = \{b_1, b_2, b_3\}$ is a basis for W in $\mathbb{R}^n$ and A is and $n \times n$ matrix with $det(A) \neq 0$, then

$$AB = \{Ab_1, Ab_2, Ab_3\}$$

is also a basis for W.