Theorem (4)**.** *If A is an $m \times n$ matrix, the range of the corresponding linear transformation is the column space of A. By Theorem (3) above, it is a subspace of $\mathbb{R}^n$.*

$T(3,4) = (3,4,7)$

Example. *The linear transformation given earlier by $T(x,y) = (x, y, x+y)$ has matrix representation*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{matrix} T(1,0) = (1,0,1)^T \\ T(0,1) = (0,1,1)^T \end{matrix}$$

Find the range of T by computing the column space of A and decompose the solution into the span of a set of vectors.

Range(T) = Col(A) = span of the columns of the matrix.

$\bar{x} = (x,y)$

$$A\bar{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} = \underbrace{x\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$$

b/c x & y are free scalar vars, this is the span of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$ is in the col(A).

In particular

$$3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

An important question in engineering, computer graphics, and many other applications is whether a given vector is in the range of a linear transformation. Now we see that this question reduces to finding solutions to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

Example.

$$T(x, y, z) = (x + 2y + z, x - y + 5z, -x + 3y + 9z)$$

A = std matrix for T

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & -1 & 5 & 4 \\ -1 & 3 & 9 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 14 & 5 \\ 0 & 5 & 10 & 4 \end{pmatrix} \quad \begin{matrix} R_2 + R_3 \\ R_1 + R_3 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -18 & -6 \\ 0 & 5 & 10 & 4 \end{pmatrix} \quad -2R_2 + R_3$$

Determine whether $\vec{\mathbf{u}} = (3, 4, 1)$ *is in the range of* T.

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -18 & -6 \\ 0 & 0 & 100 & 34 \end{pmatrix} \quad -5R_2 + R_3$$

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -18 & 6 \\ 0 & 0 & 1 & .34 \end{pmatrix}$$

yes. it's in Range.

Determine whether $\vec{\mathbf{u}} = (13, 4, 2)$ *is in the range of* T.

$$\begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$x = 7$
$y = 3$
$z = 0$

$$\begin{pmatrix} 13 \\ 4 \\ 2 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 4 \\ 2 \end{pmatrix}$$

yes

Compute the kernel of T.

$Ax = 0$

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 5 \\ -1 & 3 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & -4 \\ 0 & 5 & 10 \end{pmatrix} \quad \begin{matrix} R_1 - R_2 \\ R_1 + R_3 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -18 \\ 0 & 5 & 10 \end{pmatrix} \quad -R_3 + 2R_2$$

$$\to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -18 \\ 0 & 0 & 100 \end{pmatrix} \quad -5R_2 + R_3$$

ker(T) = sol'n to $Ax = 0$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -18 \\ 0 & 0 & 1 \end{pmatrix}$$

$z = 0$

$y - 18(z) = 0 \quad y = 0$

$x + 2(0) + 1(0) = 0$

$x = 0$

Trivial kernel

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ represents $T(x, y, z) = (x, y, 0)$

$\boxed{\text{kernel}(T) = \{(0, 0, t) = t(0, 0, 1)\}.}$

$\boxed{z = t, \quad x = y = 0.}$

A vector in the range: $(1, 2, 0)$

$T(1, 2, 3) = (1, 2, 0)$

$T(1, 2, 5) = (1, 2, 0)$

Produce infinite solutions to $Ax = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

You know $A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

& $A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$A\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = A\begin{pmatrix} 1 \\ 2 \\ 3+t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Fact: If there's a non-trivial kernel, and a solution to $Ax = b$.

Then there's infinitely many solutions to $Ax = b$.

Theorem: The range of a linear transformation ~~T:R^n -> R^m~~ $T: \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of ~~R^m~~ $\mathbb{R}^m$

Range of $T = \left\{ \begin{array}{l} \text{all } \bar{u} \in \mathbb{R}^m \\ \text{such that} \\ \quad \bar{u} = T(\bar{v}) \text{, when } \bar{v} \in \mathbb{R}^n \end{array} \right\}$

$\underbrace{\text{Let } \bar{u}_1, \bar{u}_2 \in \text{Range,}}$ show $\bar{u}_1 + \bar{u}_2 \in$ Range & $\alpha \bar{u}_1 \in R$

$\bar{u}_1 = T(\bar{v}_1)$ & $\bar{u}_2 = T(\bar{v}_2)$, $\bar{v}_1, \bar{v}_2$ live in domain $\mathbb{R}^n$

$T(\bar{v}_1) + T(\bar{v}_2) = T(\bar{v}_1 + \bar{v}_2)$, $v_1 + v_2$ lives in domain. Done.

$\alpha \bar{u}_1 = \alpha T(\bar{v}_1) = T(\alpha \bar{v}_1)$, $\alpha \bar{v}_1$ lives in domain.

Exercise: Give a relationship between the kernel of a linear transformation and the eigenvalues of the matrix that represents it?

Ex. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to$ kernel $= \bar{0}$, eigenvalues $= 1.$ (only 1).

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to$ kernel $= \{(0,0,z), z \in \mathbb{R}\}$, eigenvalues $\{1, 1, 0\}$

non-trivial

Exercise: What is the relationship between the kernel of a transformation and the reduced row echelon form of the matrix that represents it?

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$ row of zeros $\Rightarrow$ kernel non-trivial.