Linear Transformations (Linear Operators : $\mathbb{R}^n \to \mathbb{R}^m$)

Satisfy two basic properties: homogeneity and additivity

If $f: \mathbb{R}^n \to \mathbb{R}^m$ is linear then

$$f(\bar{u} + \bar{v}) = f(\bar{u}) + f(\bar{v}) \quad \text{additivity}$$

$$f(c\bar{v}) = c \cdot f(\bar{v})$$

Examples & Non Examples: can you see why the non-examples fail to meet the definition?

$$f(x,y,z) = (x+y, 2x) \qquad (\text{here}: f: \mathbb{R}^3 \to \mathbb{R}^2)$$

$$f\big((x_1, y_1, z_1) + (x_2, y_2, z_2)\big) = f(x_1+x_2, y_1+y_2, z_1+z_2)$$

$$= (x_1+x_2 + y_1+y_2,\ 2(x_1+x_2))$$

$$\stackrel{?}{=} f(x_1, y_1, z_1) + f(x_2, y_2, z_2) =$$

$$(x_1+y_1, 2x_1) + (x_2+y_2, 2x_2)$$

$$= (x_1+y_1+x_2+y_2,\ 2x_1+2x_2)$$

addition commutes / factor

equal

$$f(c(x+y), c(2x)) = (cx+cy, 2cx) = c(x+y, 2x)$$
$$= cf(x,y,z)$$

$$f(x,y) = x^2 + y$$

$$f((x_1,y_1)+(x_2,y_2)) = f(x_1+x_2, y_1+y_2) = (x_1+x_2)^2 + y_1+y_2$$

$$= x_1^2 + 2x_1x_2 + x_2^2 + y_1+y_2$$

$$\neq f(x_1, y_1) + f(x_2, y_2)$$

Section 6.2 :: **Geometry of Linear Operators** :: Math 211

(Warm-up) Why does every linear transformation map $\bar{\mathbf{0}}$ to $\bar{\mathbf{0}}$, i.e., satisfy $T(\bar{\mathbf{0}}) = \bar{\mathbf{0}}$?

$$\underbrace{T(0\cdot\bar{u})}_{= T(\bar{0})} = 0\,(T(\bar{u})) = \bar{0}$$

Also:

$$T(\underbrace{\bar{u} + (-\bar{u})}_{\bar{0}}) = T(\bar{u}) + T(-\bar{u}) \quad \text{additivity}$$
$$= T(\bar{u}) - T(\bar{u}) \quad \text{homog.}$$
$$= T(\bar{0}) = \bar{0}$$

Theorem. *Every matrix defines a linear transformation. Below, we list some basic classes of transformations.*

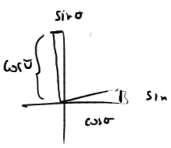
1. Norm-preserving transformations. (distance preserving)

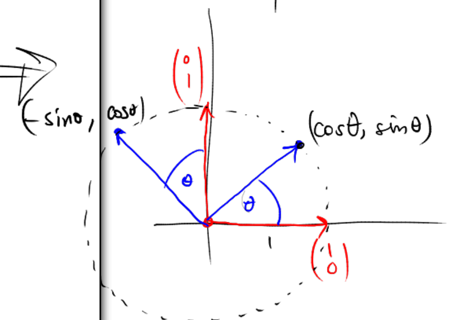
Rotation & Reflection

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Recall: when a matrix represents a transformation, its columns are the images of std. unit vectors under the transformation.



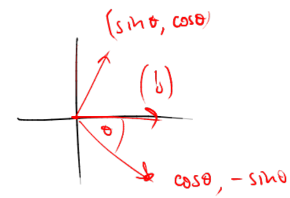


$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

2. Angle-preserving transformations.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

scales length, preserves angle



Clockwise $\curvearrowright$ $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

3. A square matrix A is *orthogonal* if $A^{-1} = A^T$. (or equivalently, $AA^T = I$)

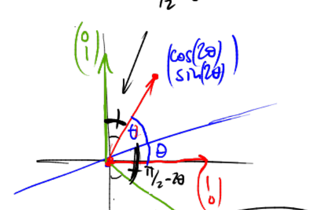
4. Examples:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad C = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad D = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Reflection across line w/ angle θ

$$AA^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta \overset{=1}{} & 0 \\ 0 & 1 \end{pmatrix}$$

$\pi/2 - \theta$



$$\frac{\pi}{2} - 2\theta + \theta = \frac{\pi}{2} - \theta$$

$$\begin{pmatrix} \sin(2\theta) \\ \cos(2\theta) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{2} - 2\theta\right) \\ -\sin\left(\frac{\pi}{2} - 2\theta\right) \end{pmatrix}$$

double angle formula

$A^{-1} = A^T$

5. Here are some of the interesting properties of orthogonal matrix A:

(btw: these transformations are rotations, reflections.

(a) The inverse, transpose, and product of orthogonal matrices is orthogonal. Can you see why?

Inverse: $\boxed{\text{Assume } A^{-1} = A^T}$, show $(A^{-1})^{-1} = (A^{-1})^T$, so $A^{-1} \cdot (A^{-1})^T$ $\| $ $A^T (A^T)^{-1} = I$

or $\downarrow$

$(A^T)^{-1} = (A^{-1})^T$ true for Ch 2

$A^{-1} = A^T$, $B^{-1} = B^T$

Assume A, B are orthogonal

show AB is orthogonal. show $(AB)^T = (AB)^{-1}$

$(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1}$

(b) $det(A) = \pm 1$.

$A^{-1} = A^T$. $AA^T = I$

so $\underbrace{\det(AA^T)} = \det(I) = 1$

$(\det(A))^2 = \det(A)\underbrace{\det(A^T)}_{\det(A)} = 1 \Rightarrow \det(A) = \pm 1$

(c) The column (row) vectors of A are orthogonal.

Fact: $AA^T = I$

If $i \neq j$ then

$(\text{row}(i) \text{ of } A) \cdot (\text{col}(j) \text{ of } A^T) = 0$

$(\text{row}(i) \text{ of } A) \cdot (\text{row}(j) \text{ of } A) = 0$

$\Rightarrow$ rows are orthogonal

(d) $||A\boldsymbol{x}|| = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

This says: A preserves norm.

$A\bar{x}$ is a vector. $||\bar{u}||^2 = u \cdot u$

$||A\bar{x}|| = \sqrt{A\bar{x} \cdot A\bar{x}} = \sqrt{\bar{x} \underbrace{A^T A}_{I} \bar{x}} = \sqrt{\bar{x} \cdot \bar{x}} = ||x||$