Section 6.3 :: **Kernel & Range** :: Math 211

Objective: Find the set of vectors that are mapped into zero by a linear transformation and use this to understand the geometry of the transformation.

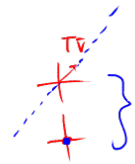
KERNEL OF A LINEAR TRANSFORMATION

If $\vec{\mathbf{x}} = t\vec{\mathbf{v}}$ is a line through the origin in $\mathbb{R}^n$ and T is a linear transformation of $\mathbb{R}^n$, there are two possibilities for the image of the line under T

$$T(\vec x) = T(t\vec v) = t(\underbrace{T(\vec v)}_{\text{fixed not changing}})$$

(1) $T(\vec v) \neq 0$ — Line $\mapsto$ Line

(2) $T(v) = \vec 0$ — Line $\mapsto$ $\vec 0$.



Likewise, if $\vec{\mathbf{x}} = s\vec{\mathbf{u}} + t\vec{\mathbf{v}}$ is a plane through the origin there are three possibilities for the image of the plane under T.

$\boxed{\text{s \& t are free variables}}$

$$T(su + tv) = T(su) + T(tv)$$

$$= s(\underbrace{T(\vec u)}_{\text{fixed}}) + t(T(\vec v))$$

when $s = 0$, $t = 0$ ← $= \vec 0$

3 possibilities = { plane thru $\vec 0$, line thru $\vec 0$, origin

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$\underbrace{\text{vectors in domain killed by } T. \ (\to \bar{0})}_{\text{kernel of } T}$

$\begin{matrix} x \\ y \\ z \end{matrix} \to \begin{matrix} 2x \\ 1 \cdot y \\ 0 \cdot 0 \end{matrix}$

line thru $\bar{0}$ in $\mathbb{R}^3$

$T(x,y,z) = (2x, y, 0)$, $\ker(T) = t(0,0,1)$ (line thru origin)

$T(x,y) = (2y, 2x, 3x-2y)$

$\left.\begin{matrix} 2y = 0 & y = 0 \\ 2x = 0 & x = 0 \\ 3x - 2y = 0 & \end{matrix}\right\} \ker(T) = (0,0)$ (trivial kernel)

$\bar{0} \in \mathbb{R}^2$

$T(x,y,z) = (\tfrac{1}{2}x, 2y, \tfrac{1}{2}z)$ $\ker(T) = (0,0,0) = \bar{0} \in \mathbb{R}^3$

$T(x,y,z) = 3x + 2y - z$

this maps $\mathbb{R}^3 \longrightarrow \mathbb{R}^1$

$3x + 2y - z = 0$

solve this.

$$\begin{pmatrix} 3 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \left.\begin{matrix} y = t \\ z = s \end{matrix}\right\} \quad \begin{matrix} 3x + 2t - s = 0 \\ x = \dfrac{s - 2t}{3} \end{matrix}$$

$$\bar{x} = \begin{pmatrix} \frac{s-2t}{3} \\ t \\ s \end{pmatrix} = \begin{pmatrix} s/3 \\ 0 \\ s \end{pmatrix} + \begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix} = s\begin{pmatrix} 1/3 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2/3 \\ 1 \\ 0 \end{pmatrix}$$

kernel is two-dimension

PLANE

thru $\bar{0}$.

Definition. *If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the set of vectors in $\mathbb{R}^n$ that T maps to $\mathbf{0}$ is the **kernel** of T, denoted $ker(T)$*

$n=3$
$m=7$

Example. *Find the kernel of*

(a) $\boxed{T_0(\vec{\mathbf{x}}) = \vec{\mathbf{0}}}$

$$T_0\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\ker(T_0) = \mathbb{R}^n$$

solve

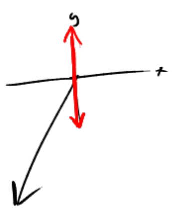
$$\bar{x} = \bar{0}.$$

(b) $T_I(\vec{\mathbf{x}}) = \vec{\mathbf{x}}$

$$\ker(T_I) = \bar{0}.$$

(c) The projection of $\mathbb{R}^3$ onto the y-axis $T(x, y, z) = (0, y, 0)$. $=(0, 0, 0)$

$x = t$
$z = s$



$$\ker(T) = (t, 0, s).$$
$$= t(1,0,0) + s(0,0,1)$$

(d) A rotation about the orgin by angle θ , with standard matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Solve Me

$$\underbrace{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}}_{I}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

BTW: $A\bar{x} = \bar{b}$ vs $A\bar{x} = \bar{0}$

sols are translation of subspace

sol's are subspaces

Notice that each of the kernels above is a subspace. In fact:

Theorem (1)**.** *For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ ker(T) is a subspace of R^n.*

Call ther $\ker(T) = S$.

Assume $\bar{u}, \bar{v} \in S$.

Show $\bar{u} + \bar{v} \in S$

$\alpha\bar{u} \in S$ $\boxed{\forall \alpha \in \mathbb{R}.}$ for all

$\bar{u} \in S$ means $T(\bar{u}) = \bar{0}$

$\bar{v} \in S$ means $T(\bar{v}) = \bar{0}$

$T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v}) = \bar{0} + \bar{0} = \bar{0}$

$T(\alpha\bar{u}) = \alpha T(\bar{u}) = \alpha\bar{0} = \bar{0}$

You have computed the kernel many times before (without knowing it) since every linear transformation has a matrix representation.

Theorem (2)**.** *If A is an $m \times n$ matrix, the kernel of the corresponding linear transformation is the solution space of $A\vec{\mathbf{x}} = \mathbf{0}$.*

Example. *The matrix representing the orthogonal projection onto the xz-plane is*

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

What is the kernel of the transformation associated to M?

$$\ker = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$$

There are often several names for the same mathematical object, each representing a different point of view. Here we have a prime example, the *kernel* of a transformation, the *solution space* of a linear system $A\vec{\mathbf{x}} = \mathbf{0}$, and the *null space* of a matrix A.

ker(T) , sol'n to $A\bar{x} = \bar{0}$, null space of matrix.

all same

RANGE OF A LINEAR TRANSFORMATION

We began by showing that lines through the origin are mapped by a linear transformation to either lines through the origin or the origin itself, and that planes through the origin are mapped to either planes through the origin or lines through the origin or the origin. This is true in general:

Theorem (3)**.** *Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps subspaces of R^n to subspaces of $\mathbb{R}^m$.*

$(-t, 4t)$

Example. *Find the image of the line $t(-1, 4)$ under $T : \mathbb{R}^2 \to \mathbb{R}^3$ where*

$$T(x, y) = (x, y, x + y)$$

$$= (-t, 4t, -t + 4t)$$

$$= t(-1, 4, 3)$$

domain / target space

Definition. *$T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, the* ***range*** *is the set of all vectors in $\mathbb{R}^m$ that are images of a least one vector in $\mathbb{R}^n$. In other words,* ***ran(T)*** *is the image of the domain of under T.*

Example. *Find the range of*

(a) $T_0(\vec{\mathbf{x}}) = \mathbf{0}$ Range $= \vec{0}$

(b) $T_I(\vec{\mathbf{x}}) = \vec{\mathbf{x}}$ Range $=$ any $\vec{x} \in \mathbb{R}^m$ $(\mathbb{R}^m)$

(c) *The projection of $\mathbb{R}^3$ onto the y-axis.* Range $=$ y-axis in $\mathbb{R}^3$

(d) *A rotation about the orgin by angle θ.*

of $\mathbb{R}^3$ range $= \mathbb{R}^3$.

$T(x, y, z) = (x, y, z, x + y)$

$T(x,y) = (2x, y)$

$$T(x_1+x_2, y_1+y_2) = (2(x_1+x_2), y_1+y_2)$$

show $T(\underbrace{(x_1, y_1) + (x_2, y_2)}_{})$

$$= T(x_1, y_1) + T(x_2, y_2)$$

$$(2x_1, y_1) + (2x_2, y_2) = (2x_1 + 2x_2, y_1 + y_2)$$

& $T(\alpha(x_1, y_1)) = \alpha T(x_1, y_1)$