1. Use the algorithm to find whether the matrix is invertible, and if possible, find the inverse. **A row of zeros on the left side of the inversion algorithm indicates the matrix is not invertible**.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 5 \\ 7 & 8 & 10 \end{bmatrix}$$

$$\begin{array}{cccccc} -7 & -14 & 0 & -7 & 0 & 0 \end{array}$$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 5 & 5 & 0 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 5 & -4 & 1 & 0 \\ 0 & -6 & 10 & -7 & 6 & 1 \end{bmatrix}$$

stop here is OK,

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 5 & -4 & 1 & 0 \\ 0 & -3 & 5 & -3.5 & 0 & .5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 5 & -4 & 1 & 0 \\ 0 & 0 & 0 & -.5 & 0 & -.5 \end{bmatrix}$$

row of 0's on 1st part $\Rightarrow$ not invertible

goal

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \end{array}\right]$$

ID

#'s

2. Use the algorithm to find whether the matrix is invertible, and if possible, find the inverse.

$$\begin{array}{cccccc} 7 & -14 & 7 & 7 & 0 & 0 \\ \end{array}$$
$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -7 & -1 & 24 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -15 & 31 & 7 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 15 & -30 & 0 & 15 & 0 \\ 0 & -15 & 31 & 7 & 0 & 1 \end{bmatrix} \quad 15R_2$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 7 & 15 & 1 \end{bmatrix} \quad \begin{array}{l} \tfrac{1}{15}R_2 \\ R_2 + R_3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -6 & -15 & -1 \\ 0 & 1 & 0 & 14 & 31 & 2 \\ 0 & 0 & 1 & 7 & 15 & 1 \end{bmatrix} \quad \begin{array}{l} -R_3 + R_1 \\ 2R_3 + R_2 \end{array}$$

$$62 - 15 = 47$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \; \underbrace{\begin{array}{ccc} 22 & 47 & 3 \\ 14 & 31 & 2 \\ 7 & 15 & 1 \end{array}}_{\text{the inverse matrix.}}\right]$$

3. Solve the system of equations

$$\begin{aligned} 1x - 2y + 1z &= -7 \\ y - 2z &= 4 \\ -7x - \cancel{8}1y + 24z &= 21 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ -7 & -1 & 24 \end{bmatrix}}_{M} \overbrace{\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\bar{x}}}^{\bar{x}} = \overbrace{\underbrace{\begin{bmatrix} -7 \\ 4 \\ 21 \end{bmatrix}}_{\bar{b}}}^{\bar{b}}$$

$$M^{-1} = \begin{bmatrix} 22 & 47 & 3 \\ 14 & 31 & 2 \\ 7 & 15 & 1 \end{bmatrix}$$

$$M \cdot \bar{x} = \bar{b}$$

$$M^{-1} \cdot M \cdot \bar{x} = M^{-1} \bar{b}$$

$$\bar{x} = \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{pmatrix} 22 & 47 & 3 \\ 14 & 31 & 2 \\ 7 & 15 & 1 \end{pmatrix} \begin{pmatrix} -7 \\ 4 \\ 21 \end{pmatrix}$$

$$= \begin{bmatrix} 97 \\ 68 \\ 32 \end{bmatrix}$$

It will be very important in this class to understand this Theorem, which can be proved using these type of elementary operations.

Theorem. *If* $\mathbf{A}$ *is a* $n \times n$ *matrix, the following are equivalent*

(a) $rref(\mathbf{A}) = \mathbf{I}$

(b) $\mathbf{A} = \mathbf{F}_1\mathbf{F}_2 \cdots \mathbf{F}_k$ *where each* $\mathbf{F}_i$ *is an elementary matrix.*

(c) $\mathbf{A}$ *is invertible.*

(d) $\mathbf{A}\vec{\mathbf{x}} = \mathbf{0}$ *has only the trivial solution.*

(e) $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is consistent for every vector* $\vec{\mathbf{b}} \in \boldsymbol{R}^n$.

(d) $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *has exactly one solution every vector* $\vec{\mathbf{b}} \in \boldsymbol{R}^n$.

1. A nonempty set of vectors in $\mathbb{R}^n$ is a **subspace** of $\mathbb{R}^n$ if it is *closed under addition and scalar multiplication.*

Plane: $x + y + z = 0$, is a subspace b/c

$\Rightarrow x = -y - z$

$= \{(x, y, z) \mid x = -y - z\}$

Two points in plane: $(-y_1 - z_1, y_1, z_1)$
$(-y_2 - z_2, y_2, z_2)$

Add:
$(-y_1 - y_2 - z_1 - z_2, \; y_1 + y_2, z_1 + z_2)$

$(-(y_1+y_2) - (z_1+z_2), y_1+y_2, z_1+z_2)$
$\rightarrow$ meets criteria to be in this plane

Scalar Mult
$\alpha(-y_1 - z_1, y_1, z_1) = (-\alpha y_1 - \alpha z_1, \alpha y_1, \alpha z_1)$
has the form specified by plane.

2. The zero subspace and $\mathbb{R}^n$ itself are **trivial subspaces** of $\mathbb{R}^n$.

$Z = \{\bar{0}\}$. $\quad \bar{0} + \bar{0} = \bar{0}$
$\alpha \bar{0} = \bar{0}$

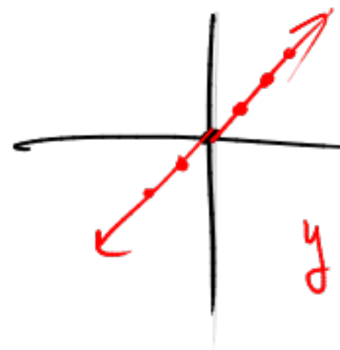
$\mathbb{R}^n$: $\quad \bar{x} \in \mathbb{R}^n$, $\bar{y} \in \mathbb{R}^n$ $\quad \bar{x} + \bar{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \in \mathbb{R}^n$ $\quad \alpha \bar{x} \in \mathbb{R}^n$.

3. Why does every subspace contain the origin? (the zero vector)

(i) 0 is a scalar & $0 \cdot \bar{v} = \bar{0}$,
since subspaces are closed $\bar{0} \in$ Subspace

or

(ii) let S = subspace. if $\bar{v} \in S$, then $-\bar{v} \in S$.
$\bar{v} + (-\bar{v}) = \bar{0} \in S$.



$\mathbb{R}^2$

$y = 3x$

x	y
0	0
1	3
-1	-3
2	6

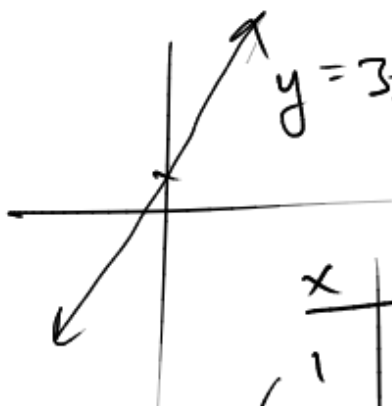
Take two, add, ask does the sum live on the line

$+$

$(3,9)$ yes. $\boxed{\overset{+}{\text{closed}}}$

$\pi(1,3) = (\pi, 3\pi)$ lives on line

$\boxed{\overset{*}{\text{closed}}}$



$y = 3x + 1$

x	y
1	4
2	7

$(3,11) \overset{?}{=}$ $11 \neq 3(3)+1$

$\boxed{\text{not } \overset{+}{\text{closed}}}$

Definition. *The* ***span*** *of* $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots \vec{\mathbf{v}}_n$ *is the set of all possible linear combinations of* $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots \vec{\mathbf{v}}_n$.

Ex.  $(1,2)$ $\quad \text{span}(1,2) = \alpha(1,2)$ line thru origin

$$\text{span}\{\bar{v}_1, \dots, \bar{v}_n\} = c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_n\bar{v}_n$$

for all possible c_i

The span of a set of vectors is always a subspace of $\mathbb{R}^n$.

$$(c_1 + d_1)\bar{v}_1 + \_\_\_ + (c_n + d_n)\bar{v}_n$$

Example: The vector $\vec{\mathbf{0}}$ spans the subspace Z $= \{0\}$.

Example: The standard unit vectors span $\mathbb{R}^3$.

$$5\begin{pmatrix}1\\0\\0\end{pmatrix} + 7\begin{pmatrix}0\\1\\0\end{pmatrix} + 8\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}5\\7\\8\end{pmatrix}$$

4. What are all the subspaces of $\mathbb{R}^2$?

the origin, lines through the origin, R^2 itself

5. What are all the subspaces of $\mathbb{R}^3$?

the origin, lines through the origin, planes through the origin, R^3 itself