
Theorem. *The solution set of* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is the translation of the solution set of* $\mathbf{A}\vec{\mathbf{x}} = \mathbf{0}$ *by any* $\vec{\mathbf{x}}_0$ *where* $\mathbf{A}\vec{\mathbf{x}}_0 = \vec{\mathbf{b}}$.

1. Find the general solution to $\boxed{\mathbf{A}\vec{\mathbf{x}} = \mathbf{0}}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = -2x_2$

$x_1 + 2x_2 = 0$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_2 - x_3 = 0$$

$x_2 = x_3$

∞ - sol's

$x_3 = t$ (free var)

Augmented Matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix}$$

$$A\bar{x} = 0 \Rightarrow \boxed{\bar{x} = \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}$$

2. Now find the general solution to $\mathbf{A}\vec{\mathbf{x}} = (0, 4, -4)$.

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 = & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_3 = t$

$1 \cdot x_2 - x_3 = -4 \Rightarrow x_2 = x_3 - 4 = t - 4$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2 = -2t + 8$

$$\bar{x} = \begin{pmatrix} -2t + 8 \\ t - 4 \\ t \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} + \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} = \boxed{t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix}}$$

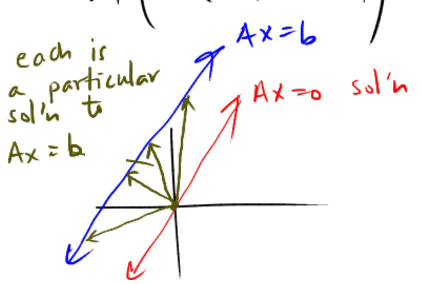
$Ax = \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix}$

sol'n to $Ax = 0$.

sol'n to $Ax = \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix}$

$$A\left(t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix}\right) = \underbrace{A\left(t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right)}_{\bar{0}} + A\begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix}$$

$$\bar{0} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix}$$



Definition. *The **column space** is the* $\underbrace{\textit{span of the column vectors of } \mathbf{A}}_{\text{set of all linear combos}}$.

Think: columns are vectors.

Ex. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\boxed{\mathrm{col}(A) = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

Theorem. *A linear system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is consistent if and only if* $\vec{\mathbf{b}}$ *is in the column space of* $\mathbf{A}$.

Think: A transforming $\bar{x}$ into $\bar{b}$.

Ex. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

1. $A\bar{x} = \bar{b}$ is not consistat
2. $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is not in span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

3. Is (0,0,1) in col($\mathbf{A}$)?

4. If $\mathbf{A}$ is an $m \times n$ matrix, then the solution space of the homogeneous linear system $\mathbf{A}\vec{\mathbf{x}} = \mathbf{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of $\mathbf{A}$.

sol'n space to $Ax = 0$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_3 = t \quad x_2 = 0 \quad x_1 = 0 \Rightarrow \bar{x} = \left\{ \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is orthogonal to each row

$\Leftarrow$ this ALways happens for $A\bar{x} = \bar{0}$.

5. Verify that particular solutions to $\mathbf{A}\vec{\mathbf{x}} = \mathbf{0}$ are orthogonal to the row vectors of $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

3.6 Special Forms

Definition. *A diagonal matrix is one whose only non-zero entries are found on the diagonal.*

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

scaling by 5 in x dir.
4 in y
3 in z

scaling by 5 in all directions.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \text{off-diagonal matrix}$$

6. Powers of diagonal matrices are easy to compute.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad A^2 = \qquad A^3 = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 125 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 25 \end{pmatrix}$$

each entry gets squared

Definition. *An upper (lower) triangular matrix is a square matrix in which all entries above (below) the diagonal are zero.*

upper triangular:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

only non-zeros appear on the diag or above

$$\begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

strictly upper Δ

lower trians.

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{pmatrix}$$

7. These matrices are used to create efficient algorithms for soving matrix equations (computing inverses, etc.)

Definition. *A symmetric matrix has the property* $\mathbf{A}^T = \mathbf{A}$ *and (skew-symmetric if* $\mathbf{A}^T = -\mathbf{A}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

symmetric

skew sym

$$A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$-A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

8. If $\mathbf{A}$ is an invertible symmetric matrix then $\mathbf{A}^{-1}$ is symmetric. Let's prove this.

Facts A is invertible. (A^{-1} exists)

A is symmetric $A = A^T$

$$(A^T)^{-1} = (A^{-1})^T$$

$$(A^T)(A^{-1})^T = (\underbrace{A^{-1}A}_{I})^T = I^T = I$$

$$\boxed{\text{show} \quad (A^{-1})^T = A^{-1}}$$

$A^T = A$ fact

$(A^{-1})^T = (A^T)^{-1}$

sub.

$(A^{-1})^T = A^{-1}$ done.