The determinant of square matrix is real number, computed using all the entries of the matrix. Every square matrix has one. In many ways they classify or determine the geometry and the solutions associated matrix equations.

Know how to compute determinants of 2x2 and 3x3 matrices by hand. Higher dimensional matrix determinants are calculated similarly. (Every entry gets used, and the det is formed by alternating sums of elementary products

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = -2$$

3x3 - two ways

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 2 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 2 & 3 & 2 \end{pmatrix}$$

$$\underbrace{1\cdot 3\cdot 2 + 2\cdot 3\cdot 2 + 1\cdot 2\cdot 3}_{6 + 12 + 6 = 24} - 2\cdot 3\cdot 1 - 3\cdot 3\cdot 1 - 2\cdot 2\cdot 2$$

$$\underbrace{-6 - 9}_{-15} - 8 = -23$$

$$= 1$$

COFACTER METHOD

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 2 & 3 & 2 \end{vmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$+$ $i+j =$ even

$-$ $i+j =$ odd

1. Pick row (or column). Row 1

$$\det = \underset{(1,1)\text{-entry}}{1}\cdot(\underset{\substack{\uparrow \\ \text{depends} \\ \text{on} \\ \text{entry}}}{+1})\begin{vmatrix} 3 & 3 \\ 3 & 2 \end{vmatrix} \underset{\substack{\uparrow \\ (1,2) \\ \text{entry}}}{-} 2\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix}$$

$(1,2)$ entry $\Rightarrow$ negative

$$= (6-9) - 2(4-6) + 1(6-6)$$

$$= -3 - 2(-2) = -3 + 4 = 1$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 3 & 1 & 5 \end{vmatrix} = -4\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} + 0\underbrace{\begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix}}_{\emptyset} - 0\underbrace{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}}_{\emptyset} \rightarrow \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Row 2 $= -4(10-3) = -4(7) = -28$

If a matrix has a row or column of all zeros, its determinant is zero.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} \qquad \det = \emptyset$$

$\emptyset + \emptyset + \emptyset - \emptyset - \emptyset - \emptyset$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{pmatrix}$$

Row 3 = Row 2 + Row 1

If the rows (or columns) of a matrix are linearly dependent, the determinant of the matrix is zero.

$$1 \cdot \begin{vmatrix} 5 & 6 \\ 7 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 5 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 5 & 7 \end{vmatrix}$$

$\underbrace{45-42}_{3} - 2(\underbrace{36-30}_{6}) + 3(\underbrace{28-25}_{3})$

$3 - 12 \longrightarrow 9$

$= \emptyset.$

$$\left|\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix}\right| = -4\begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 5 \\ 3 & 9 \end{vmatrix} - 6\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix}$$

$= -4(-3) + 5(-6) - 6(-3)$

$12 - 30 + 18 = \emptyset$

FACT:

$\det(M) = \det(M^T)$.

see examples above

$\det(M) = 0$ when
- M has a row/col of zeros
- The rows/cols of M are linearly dependent.

Ex Are $(1,0,1)$, $(0,1,0)$, $(1,2,3)$ L.I. or L.D.?

$$c_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

if $\exists\, c_i$ s.t. eqn is true $\Rightarrow$ L.D

else L.I.

$$\downarrow$$

$$\left|\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix}\right|$$

$\det = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 3 - 1 = 2$

$\Rightarrow$ L.Independent.

Examples: The identity matrix, 2x2 rotation matrix

$$\left|\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = 1,$$

Fact: $\left\{\begin{array}{l}\text{det of any } \Delta\text{'lar matrix (diagonal)} \\ \underline{\underline{\text{is}}} \text{ just product of diag. entries}\end{array}\right.$

$$\begin{vmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{vmatrix} = a_1 \begin{vmatrix} a_2 & 0 \\ 0 & a_3 \end{vmatrix} = a_1 (a_2 \cdot a_3)$$

$$\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

Notes about det(M)

1. How do elementary row ops affect det? $|kM| = k^n |M|$ if $M = n \times n$ matrix
2. What do a row of zeros do to det?
3. How does det(M) control if M is invertible? $\Rightarrow \left\{\begin{array}{ll} \det(M) = 0 \Rightarrow M \text{ is not invertible} & M^{-1} \text{ DNE.} \\ \det(M) \neq 0 & M^{-1} \text{ exist.} \end{array}\right.$
4. What is det(AB)?
5. What if det(ABA^{-1}) = 0 and A is invertible?

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

add rows $\Rightarrow$ preserves det.

Fact:

$$\det(AB) = \det(A)\det(B)$$

$$\det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1})$$

$$\det(A \cdot A^{-1}) = \det(I) = 1$$

$$\Rightarrow \frac{1}{\det(A)} = \det(A^{-1})$$

if A is inv.