Ex Groups of order 100

Here's some:

1. D_{50}
2. $\mathbb{Z}_{100}$
3. $\mathbb{Z}_4 \oplus \mathbb{Z}_{25}$
4. $\mathbb{Z}_{50} \oplus \mathbb{Z}_2$
5. $\mathbb{Z}_{20} \oplus \mathbb{Z}_5$
6. $D_5 \oplus \mathbb{Z}_{10}$
7. $\mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$
8. $\mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$
9. $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4$
10. $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\underset{4}{\mathbb{Z}_5} \oplus \underset{5}{\mathbb{Z}_5} \oplus \underset{2}{\mathbb{Z}_4}$ $\quad x$, $|x| = 5 = \operatorname{lcm}(|x_1|, |x_2|, |x_3|)$

$\underset{4}{\mathbb{Z}_5} \oplus \underset{5}{\mathbb{Z}_5} \oplus \underset{2}{\mathbb{Z}_2} \oplus \underset{2}{\mathbb{Z}_2}$

Claim: these groups are not isomorphic.

proof: If they are $\cong$, they have the same # of elts, of any given order.

# of elts	16	~~4~~ 24
order	5	10

# of elts	16	~~8~~ 74
order	5	10

$10 = \operatorname{lcm}(5,2,1,1)$

If $g \in \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4$ & $|g| = 10$ then

$10 = \operatorname{lcm}(|g_1|, |g_2|, |g_3|)$

w/ $g = (g_1, g_2, g_3)$

$10 = 1 \cdot 2 \cdot 5$

If $|g_1| = 1$, & $|g_2| = 1$ then no choice of g_3 will give 10 ($\operatorname{lcm}(1,1,\tfrac{1}{\substack{2\\4}}) \neq 10$)

$\Rightarrow$ we must have either g_1 or g_2 w/ order 5.

Assume $\checkmark$ $|g_1| = 5$.

- what choices $\checkmark$ for g_2?: $|g_2| = 5$ or 1 (g_2 could be any elt of $\mathbb{Z}_5$)

$\operatorname{lcm}(5,5$

for g_3?: $|g_3| = 2$ $\checkmark$ or $|g_3| = 4$ ($\times$ impossible) then $\operatorname{lcm}(5, (\substack{1\\ \text{or}\\ 5}), 2) = 10$

2×5

$\boxed{(3,4,0) \quad \text{order } 5, \text{ order } 5, \text{ order } 1 \quad \operatorname{lcm}(5,5,1) = 5}$

$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4$ ← # of elts w/ order 2 → just 1

# of elts of order 5: 4

of x w/ $|x| = 1$ or $|x| = 5$: 5

$\Rightarrow 4 \cdot 5 \cdot 1 = \boxed{20}$

repeat argument:

$g = (g_1, g_2, g_3)$, g_1: 1, g_2: 5

$2 \cdot 5 = 10 = |g| = \operatorname{lcm}(|g_1|, |g_2|, |g_3|)$

$|g_1|$: 1 or 5; $|g_2|$: 1 or 5; $|g_3|$: order 2

$(1, 2, 2)$ ← this elt doesn't appear in that list

	order 1	order 5	order 2
# of possible	1	4	1

$= 1 \cdot 4 \cdot 1 = 4$

$\Rightarrow$ 24 total

$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ # of g w/ $|g| = 10$

any such $g = (g_1, g_2, g_3, g_4)$

$10 = \text{lcm}(|g_1|, |g_2|, |g_3|, |g_4|)$

$\text{lcm}(1 \text{ or } 5, 1 \text{ or } 5, 1 \text{ or } 2, 1 \text{ or } 2)$ $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Case 1 $|g_1| = 1$, $|g_2| = 5$, $|g_3| = 1$, $|g_4| = 2$

#	1	4	1	1	$= 1 \cdot 4 \cdot 1 \cdot 1 = 4$

Case 2 $|g_1| = 1$, $|g_2| = 5$, $|g_3| = 2$, $|g_4| = 1$ or 2

#	1	5	1	2	$= 10$

case 3 $|g_1| = 5$, $|g_2| = 1$ or 5, $|g_3| = 1$, $|g_4| = 2$ 20

#	4	5	1	1

case 4 $|g_1| = 5$, $|g_2| = 1$ or 5, $|g_3| = 2$, $|g_4| = 1$ or 2

#	4	5	1	2	40

$+$

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Friday - Week 10

When is a direct product cyclic.

Prob $\langle(1,1)\rangle$ $\qquad$ $\langle(1,2)\rangle$ home $\qquad$ class $\langle(1,3)\rangle$

(non-Ex) $\mathbb{Z}_2 \oplus \mathbb{Z}_4 = \{(0,0), (0,1), (1,0), (1,1), (0,2), (0,3), (1,2), (1,3)\}$

$(0,0)$ ↓ identity

$(0,2)$ ↓ could this be a generator of $\mathbb{Z}_2 \oplus \mathbb{Z}_4$?

$(0,2)^3 = (3\cdot 0, 3\cdot 2 \bmod 4) = (0,2)$

$\langle(1,1)\rangle = \{(1,1), (1,1)^2 = (2 \bmod 2, 2 \bmod 4) = (0,2), (1,1)^3 = (3 \bmod 2, 3 \bmod 4) = (1,3)$

$(1,1)^4 = (4 \bmod 2, 4 \bmod 4) = (0,0)\}$ $\qquad 4 = \operatorname{lcm}(\underset{\mathbb{Z}_2}{\overset{2}{|1|}}, \underset{\mathbb{Z}_4}{\overset{4}{|1|}}) = 4$

$\langle(1,2)\rangle = \{(1,2), (0,0)\}$

$\langle(1,3)\rangle = \langle(1,1)\rangle$

Thm $G \oplus H$ is cyclic iff G, H are cyclic & $|G|$ & $|H|$ are rel prime.

proof: $\boxed{\Rightarrow}$ Assume $G \oplus H$ is cyclic (& G, H are cyclic) Prove $\underset{m}{|G|}, \underset{n}{|H|}$ are rel. prime ($g^m = e_G$, $h^n = e_H$)

Show $d = \gcd(m,n)$ is 1 ... $d = 1$

we know $G \oplus H = \langle(g,h)\rangle$. Consider $(g,h)^{\frac{m \cdot n}{d}} = \left(g^{\frac{mn}{d}}, h^{\frac{mn}{d}}\right)$

$\Rightarrow |(g,h)| = \boxed{k \text{ divides } \frac{mn}{d}} \Rightarrow d = 1$ $\qquad = \left((g^m)^{\frac{n}{d}}, (h^n)^{\frac{m}{d}}\right)$

$\boxed{k \le \frac{mn}{d}}$ $\qquad = \left(e_G^{\frac{n}{d}}, e_H^{\frac{m}{d}}\right) = (e_G, e_H)$

(Thm 8.1) $\underset{k}{|(g,h)|} = \operatorname{lcm}(m,n) = mn$

(ex: m & n are rel prime. $\}$ $\gcd(m,n) = 1$

$m \cdot n = \operatorname{lcm}(m,n) \cdot \underbrace{\gcd(m,n)}_{=1}$ (HW)

$\boxed{\Leftarrow}$ Assume G, H cyclic $|G| = m$, $|H| = n$, & $d = \gcd(m,n) = 1$ Show $G \oplus H$ is cyclic

Assumption $G = \langle g \rangle$, $g^m = e_G$ $\quad H = \langle h \rangle$, $h^n = e_H$.

Note: $|G \oplus H| = m \cdot n$ (always)

All we have to is find an elt. of order mn in $G \oplus H$ — then $G \oplus H$ is cyclic

$(g,h)^{mn} = (g^{mn}, h^{mn}) = (e_G, e_H) = e$

If $k \le mn$ & $(g,h)^k = e$ then $(g^k, h^k) = (e_G, e_H)$

This tells us something about k & $m = |g|$ & k & $n = |h|$.

$\Rightarrow m|k$ & $n|k$.

But m, n are rel. prime & $k \le mn$

$\Rightarrow k = 1$ or $k = mn$. $\Rightarrow |(g,h)| = mn$.

Consequence of these ideas:

<u>thm</u>: If s, t rel prime

$$U(st) = U(s) \oplus U(t)$$

<u>EX</u> $U(12) = U(3) \oplus U(4)$

$U(12) = \{1,5,7,11\}$

$U(3) = \{1,2\} \cong \mathbb{Z}_2$, $U(4) = \{1,3\} \cong \mathbb{Z}_2$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

order 2 order 2

not rel prime

Is $U(12)$ cyclic?

V_4 $\mathbb{Z}_4$

$\Rightarrow$ not <u>cyclic</u>