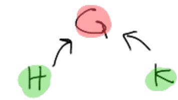
Wednesday – Week 10

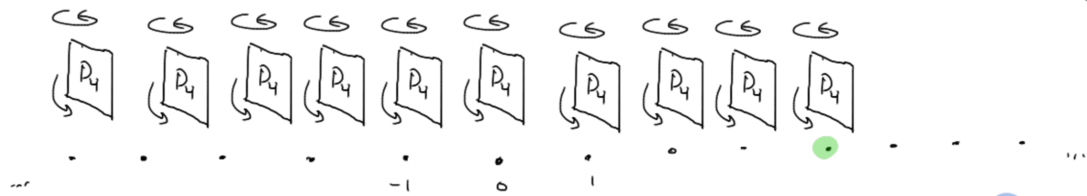
More Direct Products



$HK = \{h \cdot k\}$

Example of a direct product of groups.

$$D_4 \oplus (\mathbb{Z}, +) = \{(d, z) \mid d \in D_4,\ z \in \mathbb{Z}\}$$



$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

Properties of Direct Products:

<u>Thm</u>: The order of an element in a D. product

Let $G = G_1 \oplus G_2 \oplus \ldots \oplus G_n$

Let $g \in G$. $g = (g_1, g_2, \ldots, g_n)$

$$|g| = \operatorname{lcm}(|g_1|, |g_2|, \ldots, |g_n|)$$

proof: set $\boxed{k = \operatorname{lcm}(|g_i|)}$ $\Big\}$ $k = |g_1| m_1$, $k = |g_i| \cdot m_i$

$g_1^k = g_1^{|g_1| \cdot m_1} = e^{m_1} = e$

(1) $g^k = (g_1, g_2, \ldots, g_n)^k =$

$$\underbrace{(g_1, g_2, \ldots, g_n)(g_1, g_2, \ldots, g_n) \ldots (g_1, g_2, \ldots, g_n)}_{k}$$

$$= (g_1^k, g_2^k, \ldots, g_n^k) = (e_1, e_2, \ldots, e_n) = e$$

$\Rightarrow |g|$ divides k, ($|g| \leq k$)

(2) set $t = |g|$.

$g^t = e = (g_1^t, g_2^t, \ldots, g_n^t)$

$e = (e_1, \ldots, e_n)$ $\Rightarrow$ $\boxed{g_i^t = e_i}$

So $|g_i|$ divides t, $\boxed{t \text{ is a multiple of } |g_i|}$

$\boxed{k \leq t}$ $\Rightarrow k = \operatorname{lcm}(|g_i|) = |g|$

Ex. $\mathbb{Z}_4 \oplus \mathbb{Z}_6$ (+ mod 4, + mod 6)

$\boxed{|(2,5)| = 6}$ component-wise composition

$2 \in \mathbb{Z}_4$, $|2| = 2$

$5 \in \mathbb{Z}_6$, $|5| = 6$

$(2,5)^2 = (2,5) \cdot (2,5)$

$= (2+2 \bmod 4,\ 5+5 \bmod 6)$

$= (2(2) \bmod 4,\ 5(2) \bmod 6)$

$= (0, 4)$

$(2,5)^3 = (6 \bmod 4, 15 \bmod 6)$

$= (2, 3)$

$(2,5)^4 = (0,4)^2 = (0, 8 \bmod 6)$

$= (0, 2)$

$(2,5)^5 = (10, 25) = (2, 1)$ (% 4, % 6)

$(2,5)^6 = (12, 30) = (0, 0)$ (% 4, % 6)

Ex Groups of order 100

Here's some:

1. D_{50}
2. $\mathbb{Z}_{100}$
3. $\mathbb{Z}_4 \oplus \mathbb{Z}_{25}$
4. $\mathbb{Z}_{50} \oplus \mathbb{Z}_2$
5. $\mathbb{Z}_{20} \oplus \mathbb{Z}_5$
6. $D_5 \oplus \mathbb{Z}_{10}$
7. $\mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$
8. $\mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$
9. $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4$
10. $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

4 5 2 x

$|x| = 5$

$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4$ $= \text{lcm}(|x_1|, |x_2|, |x_3|)$

4 5 2 2

1

$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Claim: these groups are not isomorphic.

proof: If they are $\cong$, they have the same # of elts, of any given order.

$10 = \text{lcm}(5, 2, 1, 1)$

# of elts	16	40
order	5	10

# of elts	16	80
order	5	10

elts of order 5 in ...

$\mathbb{Z}_5$: all non-id elts, $\therefore$ 4

$$|g| = |G| / \gcd(|G|, g)$$

0, 1, 2, 3

3, 1, 1, 2, 7, 43

# of Elements of order 5 in

$\mathbb{Z}_{25} \oplus \mathbb{Z}_5$ (125 elts total)

let $(a,b) \in \mathbb{Z}_{25} \oplus \mathbb{Z}_5$

$|a| = {}^{25}/_{\gcd(25,a)} = 5$

5

assume

$|(a,b)| = 5 = \operatorname{lcm}(|a|,|b|)$

Case 1 $|a| = 5$, $\underbrace{|b| = 1 \text{ or } 5}_{\text{all 5 work}}$ } 20 elts

$\{5, 10, 15, 20\}$ 4 possible

Case 2 $|a| = 1$, $|b| = 5$

$\downarrow$ 1 $\downarrow$ 4 $= 4$ elts

24 elts