"Never let a mistake go to waste."

My mistake: For $\phi : G \to \bar{G}$ an isomorphism —

trying to prove $\phi(a^{-1}) = \phi(a)^{-1}$ without <u>first</u> proving ϕ sends the identity of G to the identity of $\bar{G}$.

This is foolish since $(\phi(a))^{-1}$ is <u>defined</u> as the element whose product with $\phi(a)$ is the identity in $\bar{G}$.

We need to know more about the inverse of $\bar{G}$ before we can prove something about $(\phi(a))^{-1}$!

So why does ϕ send the identity of G to the identity of $\bar{G}$?

Let $e, \bar{e}$ be the identities of $G, \bar{G}$ respectively.

For any $a \in G$ we know

$$\phi(a) = \phi(a \cdot e) = \phi(a)\phi(e)$$

Now multiply by $\phi(a)^{-1}$

$$\underbrace{\phi(a)^{-1} \cdot \phi(a)}_{\substack{\text{must be } \bar{e} \\ \text{since these} \\ \text{elements are} \\ \text{inverses in } \bar{G}.}} = \phi(a)^{-1}\phi(a) \cdot \phi(e)$$

$$\bar{e} = \bar{e} \cdot \phi(e) = \phi(e).$$

So ϕ sends e to $\bar{e}$.

Now proving $\phi(a^{-1}) = \phi(a)^{-1}$ is easy.

$$\phi(e) = \phi(a \cdot a^{-1}) = \phi(a) \cdot \phi(a^{-1})$$

$$\phi(e) = \bar{e}$$

so $\bar{e} = \phi(a)\phi(a^{-1})$

$$(\phi(a))^{-1} \cdot \bar{e} = \underbrace{\phi(a)^{-1} \cdot \phi(a)}_{\bar{e}} \phi(a^{-1})$$

$$\phi(a)^{-1} = \phi(a^{-1}).$$

Sorry for the confusion today, but that mistake gave us a chance to learn how fundamental the identity is — the definition of the inverse <u>depends</u> on it. (& theorems!)