Friday - Week 12

(showcases relationship b/w a group & its factor group)

Thm. $G/Z(G)$ theorem:

If $G/Z(G)$ is cyclic then G is abelian.

proof: Goal: G is abelian.
This is equivalent to $Z(G) = \underline{G}$

We'll prove that $G/Z(G)$ is trivial, i.e.,

$Z(G)$ = center of group
= set of $g \in G$ that commutes with everything

$$G/Z(G) = Z(G)$$

elements are cosets

just this one coset (the identity coset.)

By assumption let $G/Z(G) = \langle g Z(G) \rangle$ for some fixed arbitrary $g \in G$

Let $a \in G$ be arbitrary in G. then

$aZ(G) \in G/Z(G)$ so $\boxed{a Z(G) = (g Z(G))^i = g^i Z(G).}$

$\{\boxed{g^i z_k}, g^i z_{k_1}, g^i z_{k_2}, \dots\}$

$gz \cdot gz \cdot gz = g^3 z$

$\{\boxed{az_1}, az_2, az_3, \dots\} = \{\boxed{x_1}, x_2, \dots, x_k\}$ set!

So $a z_1 = g^i z_2$ for some $z_1, z_2 \in Z(G)$

z_1^{-1} z_1^{-1}

$\Rightarrow a = g^i \underbrace{z_2 \cdot z_1^{-1}}_{\text{subgroup}} = g^i z$ for some $z \in Z(G)$

For $G = \mathbb{Z}$
$H = 2\mathbb{Z} = \{0, \pm 2, \pm 4, \pm 6, \dots$

$1 + H = 3 + H$
means $1 + h_1 = 3 + h_2$
for some $h_1, h_2 \in 2\mathbb{Z}$
$h_1 = 2$
$h_2 = 0$ ✓

Fact: $z \in Z(G)$ so z commutes w/ g from above.

g^i commutes w/ g above.

$$g^i \cdot g = g \cdot g^i$$

$4 + H = 6 + H = H$

$\{4, 4\pm 2, 4\pm 4, 4\pm 6, \dots\}$

So $ag = g^i(zg) = g^i(gz) = (g^i g)z = g(g^i z) = ga$

$\Rightarrow$ a was arbitrary $\Rightarrow$ Everything commutes with g.

$g \in \underline{Z(G)}$.

identity ($\approx \langle e \rangle = e$

$$G/Z(G) = \langle g Z(G) \rangle = \langle Z(G) \rangle$$

so $G/Z(G) = Z(G) = \bar{e} \Rightarrow G = Z(G)$

Thm $\boxed{G/Z(G)} \cong \mathrm{Inn}(G)$.

$\mathrm{Inn}(G)$ = set of automorphisms of G that act by conjugation:

if $\psi \in \mathrm{Inn}(G)$ then $\psi(g) = x g x^{-1}$ for some fixed $x \in G$

proof: define $\varphi : G/Z(G) \longrightarrow \mathrm{Inn}(G)$ by

$\varphi(aZ) = \psi_a$ so $\psi_a(g) = a g a^{-1}$

if $\boxed{aZ} = \boxed{bZ}$ means $aZ_1 = bz_2$ for some $z_i \in Z(G)$

$a = b z_2 z_1^{-1} = bz$ $\quad \underbrace{z \in Z(G)}_{z \text{ commutes}}$

$$\begin{array}{ccc} aZ & = & bZ \\ \downarrow \varphi & & \downarrow \varphi \\ \boxed{\psi_a} & & \boxed{\psi_b} \end{array}$$

these should be the same map.

$\psi_a(x) = a x a^{-1}$

$= (bz) x (bz)^{-1} = b z x z^{-1} b^{-1}$

$= b x \underbrace{z z^{-1}}_{e} b^{-1}$

$= b x b^{-1} = \psi_b$

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remainder of proof see text:

Thm $\boxed{G/Z(G)} \cong \mathrm{Inn}(G)$.

So the set of conjugations in $D_4 \cong V_4$.

$\boxed{\psi_{180} = \psi_{r_0}}$ b/c $r_{180} h r_{180}^{-1} = h r_{180} \cdot r_{180}^{-1} = h = \psi_0$

$\psi_h(v) = hvh^{-1} = hvh = r_{180}h = hr_{180} = v = \psi_v(v)$

$\boxed{\psi_h = \psi_v} \to \boxed{\psi_{r_{90}} = \psi_{r_{270}}}$ $\boxed{\psi_d = \psi_{d'}}$

Ex. $G = D_4$.

$Z(D_4) = \{r_0, r_{180}\}$

↳ commutes w/ all rotations (obvious)

$r_{180}h$ $\qquad hr_{180}$

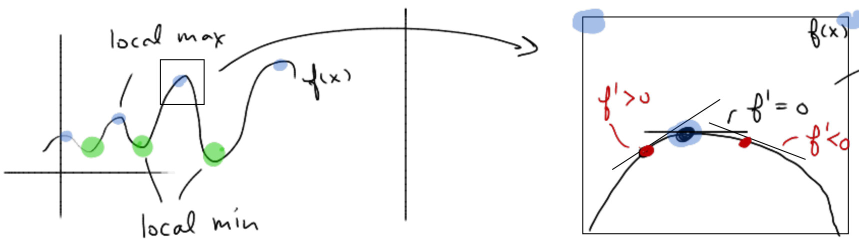
$d r_{90} d = d' r_{90} d'$

$r_{90} v r_{270} = r_{270} v r_{90}$

$$D_4/Z(D_4) \cong D_4/\{r_0, r_{180}\} \cong V_4$$

Friday - Week 12

Optimization: How to tell when you reach a Max / min.



Notice f' is changing,
going from $+ \to 0 \to -$
f' is decreasing it's derivative is negative
$\Rightarrow f''<0$

First Derivative Test	2nd Derivative Test
(1) find where $f'(x) = 0$	(1) find where $f'(x) = 0$

First Derivative Test

(2) local max: moving left to right

go from $f'>0$ to $f'=0$ to $f'<0$

$+ \to 0 \to -$

local min: local min @ $x=c$ if

$f'(c) = 0$

$f'(c-\epsilon) < 0$, $f'(c+\epsilon) > 0$

ϵ: small positive

f' $-$ 0 $+$

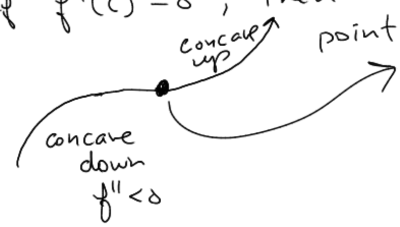
2nd Derivative Test

(2) Suppose $f'(c) = 0$

local max: $f''(c) < 0$ (concave down)

local min: $f''(c) > 0$ concave up

If $f''(c) = 0$, then c is a point of inflection.
($\Rightarrow c$ gives neither max or min)



1st Derivative Test:

Question: find the maximum & minimum values of $f(x)$ b/w $[-1, 5]$

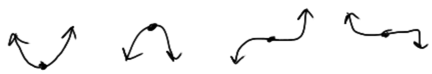
$$f(x) = x(x-4)(x+4).$$

① Find $f'(x)$, set $= 0$, solve

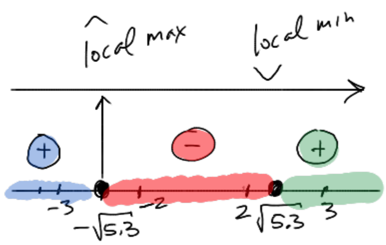
$$f'(x) = x^2 - 16 + x^2 + 4x + x^2 - 4x = 3x^2 - 16$$

$$\boxed{f'(x) = 0} = 3x^2 - 16$$

so $16 = 3x^2$, $x^2 = \frac{16}{3} = 5.\bar{3}$



$\boxed{x = \pm\sqrt{5.3}}$ critical points

② 

$$\boxed{f'(x) = 3x^2 - 16}$$

blue $f'(-3) = 3(-3)^2 - 16 = 11 > 0$

$f'(1000000) = 3(1000000)^2 - 16 > 0$

red $f'(0) = -16$

green! $f'(4) = 3 \cdot 16 - 16 > 0$

Local Max of $f(-\sqrt{5.3}) =$ ______
occurs @ $x = -\sqrt{5.3}$

Local Min of $f(\sqrt{5.3}) =$ ______
occur @ $x = \sqrt{5.3}$

Find Absolute max/min & local max/min of

$f(x) = x^4 - 16x^2$ on $[-2,5]$

(use 2^{nd} der. test)

$f'(x) = 4x^3 - 32x = 0$

$x(4x^2 - 32) = 0$

$\boxed{x = 0}$ ← critical pts

$4x^2 - 32 = 0$

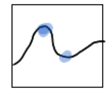
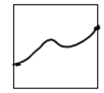
$x^2 = 8$

$\boxed{x = \pm\sqrt{8} = \pm 2\sqrt{2}}$

Absolute Max min:

occurs either:

1. @ local max/min
2. @ endpoints

①  ② 

Candidates For Abs Max	Candidates for Abs Min
$x = 0$	$\boxed{x = \pm 2\sqrt{2}}$ ✓
$x = -2$	$x = -2$
$\boxed{x = 5}$ ✓	$x = 5$
check $f(5)$ vs $f(0)$ vs $f(-2)$	Compare $f(2\sqrt{2}) = f(-2) = f(5)$

$\boxed{f''(x) = 12x^2 - 32.}$

Evaluate @ criti pts

$f''(0) = -32$ ∩ ⇒ local max @ $x = 0$

$f''(2\sqrt{2}) = 12(2\sqrt{2})^2 - 32 = 12 \cdot (8) - 32 = 64$ } $f'' > 0$ local min

$f''(-2\sqrt{2}) = 64$