① fix pre-image of $2i$

② 1^{st} Isom thm:

- D_4
- $\mathbb{Z}_n$
- $A_4 \trianglelefteq S_4$ so $f:(x) = \begin{cases} 0 & \text{even} \\ 1 & \text{odd} \end{cases}$ has ker A_4 | $\operatorname{im}(f) \cong \mathbb{Z}_2$
- Index 2 subgroup normal

  let $A = A_4$, $x \notin A$.

  $xA \neq A$ | Two cosets: xA, A } equal
  $Ax \neq A$ | Ax, A
  $\therefore$ equal

- Wrapping function

③ Homomorphisms give some properties

closely related

not $\sim S_4$ (large kernel)

- size of kernel reflects similarity $(D_4/\{r_0, r_{180}\} \cong W_4$ vs $S_4/A_4 \cong \mathbb{Z}_2$

(people study all possible homom. (G))

Isom. give alternate view of whole group

Exam 2 (Final)

- Timed _ at home, no notes, no help.
- 1 hour
- When :

    Thur : (asynchronously) _ submit Google Folder, by thurs midnight.

Friday - Week 13:

1st Isomorphism Theorem:

$f: G \to G'$, a homom

$G/\ker(f) \cong \mathrm{Im}(f) = f(G)$

$$D_4 = \{\overbrace{r_0, r_{180}}^{180}, \overbrace{r_{90}, r_{270}}^{180}, \overbrace{h, v}^{180}, \overbrace{d, d'}^{180}\}$$

$$\boxed{r_0 \quad h \quad r_{180} \quad v} = \mathrm{Im}(f) = f(D_4)$$

— given —

$f(r_{90}) = h$

$f(h) = r_{180}$

check

$f(r_{90} \cdot h) = f(d) = v$

$\| \quad f(r_{90}) f(h) = h \cdot r_{180}$

$r_{90} h = d$

$$\boxed{h \cdot r_{180} = v}$$

similarly we could check each product.

$f: D_4 \to D_4$. $\ker(f) = \{r_0, r_{180}\}$. $\mathrm{Im}(f) = \{r_0, h, r_{180}, v\}$

subgroup & normal

$$\Longrightarrow \quad D_4/\ker(f) = D_4/\{r_0, r_{180}\} \cong \{r_0, h, r_{180}, v\} \cong V_4$$

$D_4/\{r_0, r_{180}\} \cong V_4$

(cosets)

Ex. let $G = S_4$

Consider $A_4 \leq G$.

Define $\varphi : S_4 \to \mathbb{Z}_2 = \{0, 1\}$ by

$g = (123) = (13)(12)$

$$\varphi(g) = \begin{cases} 0 & \text{if } g \text{ is even} \quad (g = \text{even product of 2-cycles}) \\ 1 & \text{if } g \text{ is odd} \quad (g \neq \text{even product of 2-cycles} \end{cases}$$

$g = (12)$ ← odd # of 2-cycles

homomorphic

$$\varphi(a \cdot b) = \begin{cases} 0 & \text{if } a \text{ odd}, b \text{ odd}; \; a \text{ even}, b \text{ even} \Rightarrow ab \text{ even} \\ 1 & a \text{ even}, b \text{ odd}; \; b \text{ odd}, a \text{ even} \end{cases}$$

Ex. a even, b odd

$\varphi(ab) = 1 = \varphi(a) + \varphi(b)$

$\quad = 0 + 1$ (even)

✓

$\mathrm{Ker}(\varphi) = A_4 \Rightarrow A_4 \trianglelefteq S_4$ (normal)

1st Isom. Theorem

$\boxed{S_4 / A_4} \cong \varphi(S_4) \cong \boxed{\mathbb{Z}_2}$

Homomorphisms tell us something about the "domain" group

(size of kernel reflects how similar the two groups are)

Ex $\varphi : S_4 \to \mathbb{Z}_2$

$\Rightarrow |\mathrm{Ker}(\varphi)| = 12$, these aren't very similar.

Ex. $f_1 : \mathbb{R}^3 \to \mathbb{R}$, $f_1(x,y,z) = x$

$\mathrm{ker}(f_1) = \{(0, y, z)\}$ ← 2 degs freed

$\cong \mathbb{R}^2$ 2-dim

vs.

$f_2 : \mathbb{R}^3 \to \mathbb{R}^2$

$f_2(x,y,z) = (x, y)$

$\mathrm{ker}(f_2) = \{(0, 0, z)\}$

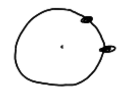
$\mathbb{R}^1$ 1-dim

Ex: $f : D_4 \to V_4$

$|\mathrm{Ker}(f)| = 2$

$\Rightarrow D_4$ & V_4 have a lot more in common

Circle Group: $H = \{a+bi \in \mathbb{C} \mid a^2 + b^2 = 1\}$

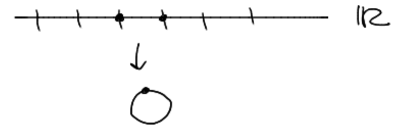


Wrapping Function: $W: \mathbb{R}^{+} \longrightarrow H^{*}$

$W(x)$ = the point x radians from $(1,0)$ on unit circle

$W(2\pi + \epsilon) = W(\epsilon)$

$W(x) = \cos x + i \sin x$



Cool: W is a homomorphism

$$W(a+b) = \cos(a+b) + i\sin(a+b)$$

$$= \cos(a)\cos b - \sin(a)\sin b + i[\sin a \cos b + \sin b \cos a]$$

$$W(a)\cdot W(b) = (\cos a + i\sin(a)) \cdot (\cos(b) + i\sin(b))$$

$$= \cos(a)\cos b + i\sin b \cos a + i \sin a \cos b - \sin a \sin b$$

$\ker(W) = \langle 2\pi \rangle \Rightarrow$ 1st Isom Thm $\mathbb{R}^{+}/\langle 2\pi\rangle \cong$ Circle group H