Homomorphisms! Structure preserving mappings of groups

Def: If G & $\overline{G}$ are groups then a homomorphism $\varphi : G \longrightarrow \overline{G}$ has the property

$$\varphi(g_1 \cdot g_2) = \varphi(g_1)\cdot\varphi(g_2).$$

Ex: $G = \mathbb{R}^*$, $\overline{G} = \mathbb{R}^*$

$\varphi(x) = x^2$ is a hom. ✓

$$\underline{\varphi(a\cdot b)} \overset{\text{def'n}}{=} (a\cdot b)^2 \overset{\text{alg}}{=} (a\cdot b)(a\cdot b) \overset{\mathbb{R}^*\text{-abelian}}{=} a^2 b^2 \overset{\text{def'n}}{=} \varphi(a)\cdot\varphi(b)$$

Id in $\mathbb{R}^*$ is 1.

$\text{Ker}(\varphi) = \{x \in \mathbb{R}^* \mid \varphi(x) = 1\}$

$= \{x \in \mathbb{R}^* \mid x^2 = 1\}$

solve $x^2 = 1$

$x = \pm 1$

$\boxed{\text{Ker}(\varphi) = \pm 1}$

lives in "domain" – not target

Ex Non-Ex: $G = (\mathbb{Z}, +)$, $\overline{G} = (\mathbb{Z}, +)$

$\varphi(x) = x^2$ is not a homomorphism

$$\varphi(a + b) \overset{\text{def'n}}{=} (a+b)^2 = a^2 + \underbrace{2ab}_{\neq 0} + b^2 \neq a^2 + b^2 = \varphi(a) + \varphi(b)$$

group op

Ex MA211: determinant of a matrix is a homomorphism.

$\det : \text{GL}(n, \mathbb{R}) \longrightarrow \mathbb{R}^*$

G = general ($\det \neq 0$)

L = linear (matrix)

n = size (square matrix)

$\mathbb{R}$ = coeffs

Fact: $\det(A \cdot B) = \det(A) \cdot \det(B)$

matrix mult. group op. in $\text{GL}_n\mathbb{R}$

real # mult. group op in $\mathbb{R}^*$

$\text{Ker}(\varphi) = \{\text{all matrices w/ } \det = 1\}$

$= \text{SL}_n\mathbb{R}$.

Ex Calculus: $\frac{d}{dx}$ is a homomorphism $\frac{d}{dx} : \mathcal{C} \longrightarrow \mathcal{C}$

$\mathcal{C}$ = ~~space~~ group of differentiable functions under addition

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right)$$

Identity in $\mathcal{C}$ is $f(x) = 0$ (b/c operation is $+$)

$\text{Ker}\left(\frac{d}{dx}\right)$ = constant functions

Properties of homomorphisms

① Preserve Identity (send $e \in G$ to $\bar{e} \in \bar{G}$).

proof: Let $g \in G$. $g^{-1} \in G$ & $e = g \cdot g^{-1}$ So.

$$\underline{\underline{\varphi(e)}} = \varphi(g \cdot g^{-1}) \underset{\substack{\text{def'n} \\ \varphi = \text{hom}}}{=} \varphi(g) \cdot \varphi(g^{-1}) \underset{[\text{see below}]}{=} \varphi(g) \cdot \varphi(g)^{-1} = \underline{\underline{\bar{e}}}$$

② Preserve inverses: $\varphi(g^{-1}) = (\varphi(g))^{-1}$.

proof

$$\varphi(g \cdot g^{-1}) = \varphi(g) \cdot \varphi(g^{-1})$$

$$\text{''}$$

$$\varphi(e) = \varphi(g) \cdot \varphi(g^{-1})$$

$$(\varphi(g))^{-1} \cdot \varphi(e) = \underbrace{\varphi(g)^{-1} \cdot \varphi(g)}_{\bar{e}} \cdot \varphi(g^{-1})$$

$$\varphi(g)^{-1} \cdot \underset{\bar{e}}{\underset{\shortparallel}{\varphi(e)}} = \bar{e} \cdot \varphi(g^{-1}) = \varphi(g^{-1})$$

$$\varphi(g)^{-1} = \varphi(g^{-1})$$

Kernel of a homomorphism:

Let $\varphi : G \longrightarrow \bar{G}$ be a homomorphism

$$\mathrm{Ker}(\varphi) \underset{def}{=} \{x \in G \mid \varphi(x) = \bar{e}\} = \{\text{set of elements sent to the id.}\}$$

Theorem: Kernel is a normal subgroup:

proof:

Subgroup: Let $k_1, k_2 \in \mathrm{Ker}(\varphi)$ w/ $\varphi : G \to \bar{G}$ is a homomorph.

Fact $\varphi(k_1) = \bar{e} = \varphi(k_2)$.

We show: $k_1 k_2^{-1} \in \mathrm{Ker}(\varphi)$:

$$\varphi(k_1 \cdot k_2^{-1}) \underset{homom}{=} \varphi(k_1) \cdot \varphi(k_2^{-1}) = \varphi(k_1) \cdot \underbrace{\varphi(k_2)}_{\bar{e}}{}^{-1}$$

$$= \bar{e} \cdot \bar{e}^{-1}$$

$$= \bar{e} \cdot \bar{e} = \bar{e}$$

So $k_1 \cdot k_2^{-1} \in \mathrm{Ker}(\varphi)$

product of arb. element & arb element

Step 1 Subgroup Test $\Rightarrow$ $\boxed{\mathrm{Ker}(\varphi) \leq G}$

Normal: Show $a k_1 a^{-1} \in \mathrm{Ker}(\varphi)$ $\forall$ $a \in G$

reminder $k_1 \in \mathrm{Ker}(\varphi)$

$$\varphi(a k_1 a^{-1}) \underset{homom.}{=} \varphi(a)\underbrace{\varphi(k_1)}\overbrace{\varphi(a^{-1})} = \varphi(a) \cdot \bar{e} \cdot (\varphi(a))^{-1} = \varphi(a)(\varphi(a))^{-1}$$

$$= \bar{e}$$

ker = anything sent to $\bar{e}$ $\Rightarrow$ $a k_1 a^{-1} \in \mathrm{Ker}(\varphi)$ $\Rightarrow$ $\boxed{\mathrm{Ker}(\varphi) \trianglelefteq G}$

Hom Props:

- id/inv.
- order: $|g|$ finite $\Rightarrow$ $|\varphi(g)|$ divides $|g|$

  let $|g| = n$. So $g^n = e$

  the $\varphi(g^n) = \varphi(e) = \bar{e} = \varphi(g)^n \Rightarrow |\varphi(g)|$ divides n $\therefore$

- $\ker(\varphi) \leq G$

  let $k_1, k_2 \in \ker(\varphi)$

  $\varphi(k_1 k_2^{-1}) = \varphi(k_1) \cdot \varphi(k_2)^{-1} = \bar{e} \cdot \bar{e}^{-1} = e$

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on Subgroups

φ preserves	subgroups cyclic abelian normal	pre-image of subgroups (normal) are subgroups (normal) 1-1, onto, homo = iso.

$|\ker(\varphi)| = n \Rightarrow \varphi$ is n-1 mapping

$\varphi(g) = g' \Rightarrow \varphi^{-1}(g') = g\ker(\varphi)$. $x \in g\ker(\varphi) \Leftrightarrow \varphi(x) = \varphi(gk) = \varphi(g)\varphi(k) = \varphi(g)\bar{e} = \varphi(g) = g'$

all cosets have same # of elts (that of $\ker(\varphi)$)